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Mathematics in Hebrew in Medieval Europe

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INTRODUCTION

This chapter covers mathematics written in Hebrew between the eleventh and sixteenth centuries in Europe. But the term "Hebrew mathematics" risks conferring an illusory unity on the corpus presented. As we shall see, the mathematical works covered in this section are strongly rooted in the non-Hebrew scientific traditions around which they evolved.

The Hebrew works are linked together by a common thread of canonized Hebrew references and sometimes are concerned with applications to Jewish philosophy or law, but these links are generally weaker than those that bind each work to its specific non-Jewish surroundings. These relatively weaker links, however, should not be ignored. Indeed, they are strong enough to compose some interesting syntheses and sometimes cast Hebrew mathematics in the role of a cultural "go-between."

The twelfth century, which is the starting point of this collection (except for a short extract from Rashi), is not the beginning of Hebrew mathematics. Indeed, one can find some mathematical discussions in the rabbinical literature as early as the *Mishnah* (100–300 CE). But these are few, far between, and do not constitute a systematized body of knowledge. Later on, Jews living in Islamic countries before the twelfth century continued writing mathematical treatises, but they used classical Arabic as their scientific language and addressed their writings to the entire mathematical community, not specifically to Jews.

It was only in the twelfth century that the demand and opportunity arose to produce original and translated Hebrew scientific works. In the first half of that century, the cultural divide between Arabophone Judaism south of the Pyrenees and Hebreophone Judaism to their north led to the composition of several scientific works in Hebrew by Arabophone Jewish scholars. This trend gained momentum the following decades. In the middle of the century, persecutions in Muslim Spain led to the emigration of Arabophone scholarly Jewish families to Christian environments, especially to southern France. In these cultural contexts Jews did not know Arabic, and the cultural language was Hebrew. Nor did they know Latin, the literary language of the majority culture. Scientific works in the different vernaculars did not yet exist, and Jews did not use them for writing. Thus, it was in these locations that Hebrew emerged as a scientific language.

The first representatives of this transition are also the most famous and most canonized writers of Hebrew mathematics: Abraham ibn Ezra and Abraham bar Ḥiyya (Savasorda). Both are witnesses of the Maghribian/Andalusian branch of Arabic practical mathematics (*muʿāmalāt*)—a branch that did not survive in its original language. Both authors link this branch of mathematics to questions of religious law, astronomy, and number theory, and both had a role in the transmission of this knowledge into Latin. A brief extract of Talmudic exegesis from Rashi, the most renowned commentator on the Bible, shows that the integration of practical measurement and biblical exegesis precedes Ibn Ezra and Bar Ḥiyya; excerpts from Simon ben Ṣemaḥ Duran's *responsa* indicate the lasting presence of this integration.

The next Hebrew mathematical corpus is the thirteenth- and fourteenth-century translations from Arabic to Hebrew executed in Provence by and around the Ibn Tibbon family. They managed to provide Hebrew readers with Hebrew versions of a rather comprehensive sample of canonical Greco-Arabic mathematical texts. Since the present collection excludes literal translations of such identified Arabic sources, we include almost nothing from this important tradition of Hebrew mathematics. Our only representatives of this circle are a short number theoretic excerpt from Qalonymos ben Qalonymos's *Book of Kings* and his translation of a treatise on polyhedra whose source is not identified. But this should not detract from the importance of the translators' work: they introduced into Hebrew an ideal of literal translation, provided an infrastructure for Hebrew science, and created a hitherto nonexistent mathematical vocabulary, coining terms either by allocating new meanings to existing Hebrew words or by introducing loan translations from Arabic.

Next follows the most original scene of Hebrew mathematicians, active in and around the Iberian Peninsula in the fourteenth and fifteenth centuries. This chapter shows the span of their work. We include some arithmetic and geometry by Levi ben Gershon (Gersonides), the most original Hebrew mathematician, whose sources have been difficult to identify. We also include some of the idiosyncratic and original efforts of Abner of Burgos (also known as Alfonso di Valladolid), who followed Archimedean traditions. We further include work by Immanuel Bonfils on circle measurement and decimal fractions (the first documented appearance of the latter on European soil) and a sample of little-known authors of practical mathematics: Jacob Canpanṭon, Aaron ben Isaac, and an anonymous source with a more algebraic flavor.

During the fourteenth century and especially the fifteenth century, numerous Jews had to emigrate from Spain and southern France due to Christian persecution, and their mathematical books and knowledge migrated along with them. Isaac ibn al-Aḥdab, representing the first generation of migrants (he migrated from Spain through the Maghrib to Sicily), carried with him the knowledge produced by commentators on the algebra of Ibn al-Bannāʾ. Later on, the Italian mainland saw Jews working with both older Hebrew versions of Arabic sources and newer Latin and vernacular sources. Our examples include Shlomo ben Isaac's treatise on the hyperbola's asymptote and Simon Moṭoṭ's algebra.

The last mathematical scene we sample is the sixteenth century Byzantine Jewish community, which adds Greek sources to the mixture of Hebrew mathematics. Mizraḥi, whose work we represent here, shows his excellent capacity as a mathematical collector and editor—so much so that his work, late and elementary as it was, was still considered worthy of a Latin translation and printing in 1546.

Since the chronological order of these works does not reflect a historical continuity, the order of presentation here follows themes and levels of complexity. We begin with arithmetic and then discuss number theory and combinatorics. We follow this with measurement theory and practical geometry, next include some highbrow scholarly geometry, and finally conclude with algebra. In each section, the presentation begins with the more elementary and proceeds to the more sophisticated.

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NOTE ON TRANSLATION AND TRANSCRIPTION

Since this chapter pools together various existing editions and translations, we could not hope to obtain a uniform translation standard. This should be borne in mind when comparing terms and forms of expression across the different translations. Almost all English translations here are based on a Hebrew manuscript or edition and not on a secondary translation.[1] Some of the translations brought over from English editions were slightly adapted or revised. For biblical quotations we used the Jewish Publication Society Bible.

Where it seemed necessary, we include explanatory footnotes in modern mathematical notation. These should be handled with care. The modern mathematical transcription may not faithfully represent the concepts and procedures reflected in the original sources.

Numbers were either represented as words or according to a key that gave each Hebrew letter a numerical value (the first nine letters represented 1–9, the next nine letters represented 10–90, and the final four letters represented 100–400). The letters were combined additively to form composite numbers, but thousands were sometimes represented apart from lower numbers, because composing too many letters was impractical. Decimal place-value representation was rarely used, and usually only in calculation diagrams or when dealing with large numbers. There, too, Hebrew authors preferred using the first nine Hebrew letters over Arabic numerals. In this volume, numbers written as words are translated as words, and those written as sign combinations are translated into the modern place-value representation with Arabic numerals. However, different manuscripts of the same text are often inconsistent, and some of the translations used here might deviate from this practice.

As for the transliteration of Hebrew names and terms, when they have a standard English transcription (e.g., Reuven, Isaac, Torah), we use that transcription. Other names and terms are transliterated according to the typical phonetic value of the letters, using b/v for hard and

[1]The exceptions are Levi ben Gershon's *Harmonic Numbers*, which is translated from the Latin edition (the Hebrew original being lost), and his commentary on the parallel postulate, translated by Victor Katz based on a French translation and checked against the Hebrew by Gad Freudenthal.

soft *bet*, k/kh for hard and soft *kaf*, p/f for hard and soft *pe*, ḥ for *ḥet*, ṭ for *ṭet*, s for both *samekh* and *sin*, ṣ for *ṣadiq*, q for *qof*, sh for *shin*, ʾ for *ʾlef*, and ʿ for *ʿa*yin. When silent or standing for vowels, *ʾlef* and *he* are transliterated accordingly. Long vowels and strong (doubled) consonants are not distinguished in the transliteration from their short and single parallels (with the exception of names where doubling has become standard, such as in Bar Ḥiyya and Ibn Tibbon).

I. PRACTICAL AND SCHOLARLY ARITHMETIC

This section combines practical and scholarly—as well as earlier and later—Hebrew expositions of arithmetic. From Ibn Ezra's foundational twelfth-century *The Book of Number*, we explore some elementary calculation techniques in decimal numbers and simple and sexagesimal fractions. We follow with a brief discussion of decimal numbers from the unedited arithmetic of a practically unknown Aaron ben Isaac, which sheds some light on the context of practical arithmetic at the time. Immanuel Bonfils (fourteenth century) then shows how to do arithmetic with decimal fractions as well, but does not thereby give up on sexagesimals. The lesser known Jacob Canpanṭon (fourteenth–fifteenth century), whose work has not yet been edited, provides a detailed discussion of irrational root extraction, citing several methods and providing well-reasoned error analysis. Elijah Mizraḥi (sixteenth-century Constantinople) remains at the level of elementary techniques, but provides them with lucid justifications. Finally, Levi ben Gershon sends us back to the early fourteenth century but presents the most scholarly, general, and abstract treatment of the arithmetic of his time, including detailed proofs and covering calculation techniques, proportions, series summation, and typical word problems.

1. ABRAHAM IBN EZRA, *SEFER HAMISPAR* (*THE BOOK OF NUMBER*)

Abraham ben Meir ibn Ezra (ca. 1089–1167)[1] was born in Tudela, ruled by the Emirate of Saragossa. During his lifetime he traveled extensively in North Africa, Spain, Italy, and France. Among his intellectual friends one can find Rabenu Tam and Judah Halevi. In Jewish circles he is a well-known classical poet and writer of biblical and Talmud exegesis, but he also wrote on astrology and ventured into astronomy (including a translation of al-Bīrūnī's commentary on al-Khwārizmī's astronomical tables), calendar studies, mathematics, medicine, grammar, and philosophy.

The Book of Number is the earliest surviving comprehensive Hebrew arithmetic from the Muslim period. It seems to have originated in about 1150 and enjoyed a wide circulation. It made a clear and recognizable impression on several later Hebrew mathematical compositions.

The book opens with methods for applying the four basic arithmetic operations to integers in decimal representation, simple fractions and sexagesimal fractions. It then provides an elementary treatment of arithmetic, geometric and harmonic ratios, and summation formulas for the series of the first n integers and squares. From there the book goes on to apply

[1] For a general overview, see [Sela, 2003]; for a catalog of his scholarly writing, see [Sela and Freudenthal, 2006].

proportions (the rule of three, but not under this name) to commercial and calendar problems. Next it treats the extraction of square roots, bundled together with some formulas relating sums of squares to squares of differences and sums. The concluding geometry section deals with the Pythagorean Theorem and some elementary rectilinear and circle measurement, mentioning Hellenistic and Indian calculations for pi.

The book uses three representations of numbers. The first is fully spelled out number words. The second is the classical Hebrew presentation, related to the Greek and Arabic sources, of expressing numbers by letters according to a fixed key, and combining these letters additively. The third representation is the Hindu-Arabic decimal system, with the nine digits replaced by the first nine Hebrew letters, and zero denoted by a circle. This third representation is used mostly in calculation diagrams, and rarely within the running text. Here the last two representations are rendered by modern numerals. Fractions too are sometimes represented as simple fractions (mostly in arithmetical context) and sometimes as sexagesimal or other *x*-imal fraction systems (mostly in astronomy related contexts).

Numbers and the decimal place-value system (from the Introduction)

In this section Ibn Ezra introduces the Hindu-Arabic decimal system and brings in some Jewish mysticism to complement the presentation. The multiplication sphere is probably Ibn Ezra's original invention.

As God almighty alone created in the upper world nine large spheres surrounding the earth, which is the lower world, and as the author of the Book of Creation [*Sefer Yeṣira*] said that the ways of wisdom are in number, letter and word [*sfar vesefer vesipur*], so the number consists of nine digits which exhaust all numbers. When they are in the first rank, they are called ones. Ten is likened to one and twenty to two. . . . And a hundred is likened to one and to ten, and two hundred is likened to twenty and to two, and so are a thousand and ten thousand the first among [decimal] multiples[2] [*rashey klalim*] for the numbers that follow, which are [marked by the Hebrew letters] א׳ י׳ ק׳ ב׳ כ׳ ר׳.[3]

This is shown by drawing a circle and writing nine numbers around it [Fig. I-1-1]. Multiply nine by itself. It is a square because its length equals its width; then you see it as it is. The square is 81, and indeed one, the first among units, is to the left of nine, and 8, which stands in the multiples position for eighty, is to the right of nine. If you multiply nine by eight the result is 72, and indeed 2 is to its left, and 7, standing for 70, is to its right. If you multiply 9 by 7 the result is 63, and indeed 3 is to its left, and 6, standing for 60, is to its right. If you multiply 9 by 6 the result is 54, and indeed 4 is to its left, and 5, standing for 50, is to its right. Since five lies in the middle of the 9 numbers, it is called "round," and goes round into itself, because its square contains five.[4] When you multiply 9 by 5 the units go round to the right and the multiples [of tens] to the left. Indeed, the result is 45 and the 5 is to the right of 9 and the multiples [of tens], which are 4 for 40, to its left. When you multiply 9 by 4 the result is 36 with 3 for thirty. When you multiply 9

[2]Multiplicity (*klal*) is used in opposition to a unity, designating a genuine number, or, more specifically in this text, a power of 10.

[3]That is, א (*alef*) stands for 1, י (*yud*) stands for 10, ק (*kuf*) for 100, and ב, כ, ר (*beit*, *kaf*, and *reish*) for 2, 20, and 200, respectively. This is the classical use of the Hebrew alphabet for representing numerals.

[4]The unit digit in 5 times 5 is again 5.

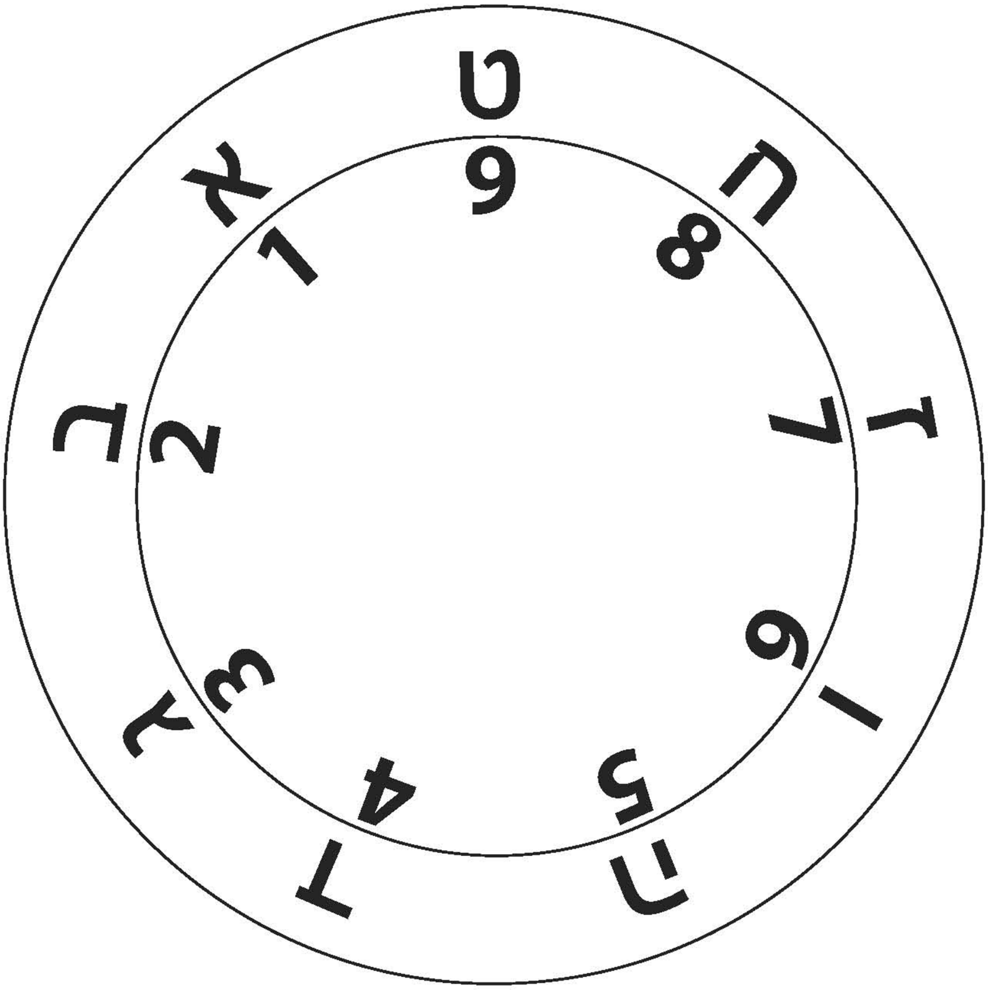


Fig. I-1-1.

by 3 the result is 27 with 2 for twenty. When you multiply 9 by 2 the result is 18 with 1 for ten. Therefore 9 serves to test the multiplication of a number by itself or by another.[5] Thus the Indian scholars based all their numbers on nine and made signs for the 9 numbers 1, 2, 3, 4, 5, 6, 7, 8, 9.[6] I designate them by the [Hebrew letters] א ב ג ד ה ו ז ח ט.[7]

If you ever have a number in the units before the position of multiples of tens, one writes first the number of units and then the number of multiples. And if there is no number in the units, and there is a number in the second rank, which is the tens, one places the image of a wheel O in the first to show that there is no number in the first rank,

[5]This refers to the verification of calculations by "casting nines" (in modern terms, verifying that the results hold when calculated modulo 9).

[6]This is the only place in the text where actual Arabic numerals are used.

[7]The letters are read *alef*, *beit*, *gimel*, *dalad*, *hey*, *vav*, *zayin*, *ḥet*, *ṭet*.

and writes the number for the tens next. And if its multiples are hundreds and tens, one writes a wheel in the first rank, then the number of tens in the second and the number of hundreds in the third. One then writes the number of thousands, if there are any, in the fourth rank, the number of tens of thousands in the fifth, and the number of hundreds of thousands in the sixth ... and so on without end. If there is a number of units and hundreds but no tens, one writes the number of units in the first rank, and a wheel in the second, and the number of hundreds in the third. Thus one keeps the ranks of the wheel according to the ranks of the number at hand, and places a wheel in the first rank, or two wheels as required at the beginning or the middle. This is the wheel O, for it is "like the whirling [wheel of] dust; as stubble before the wind" [Psalm 83:14], serving only to keep rank. In the foreign tongue [Arabic] it is called *sifra*.

The number one (from Chapter 2)

This treatment of the unit, which is in line with the Hellenistic tradition, attests to the exclusion of one from the sequence of integers.

Know that all numbers are combinations of ones, and one itself is subject to no change, multiplicity or division, but is the ground for all increase, multiplicity and division. One alone is primordial and serves to change all numbers. With its single side it affects what all numbers do with their two sides. Indeed, two precedes three on one side and four follows it on the other, and both sides combined are six which is double three, and the same goes for all numbers; while one has no preceding side, and the side that follows it has two, which is double one.

Multiplication of fractions (from Chapter 5)

This segment discusses various approaches to handling fractions. The use of a common denominator for multiplication of fractions might seem odd today, but the text sheds some light on its context.

To begin with I state the rule that the products of fractions are the opposite of the products of wholes. When one says multiply half by half, it is as if he says, take half of half, and the result is one quarter. We know that the half is taken out of two, whose half is one, and the other half is also one. The product of one by one is one, and the square of the denominator [*more*] is 4, so this last one is a quarter, which is half of half. We do the opposite of our practice with wholes, always taking the value of the product with respect to the square of the denominator. Multiplying a third by a third, the result is a ninth. Multiplying a quarter by a quarter, the result[ing denominator] is 16 and the numerator is one, which is half an eighth. Continue this way until ten and also above, as in one part in 11 multiplied by one part in 11 is one part in 121, which is the square. This way you multiply fractions of some kind by fractions of the same kind, whether equal or one bigger than the other, and then divide by the square of the multiplied denominator.

Example: We want to multiply 3 quarters by 3 quarters. The denominator is 4. For each of the 3 quarters we take 3, and the product is 9. We divide them by 16, which is the square of the denominator, and get its half and half of its eighth. If you wish you may divide 9 by 4, and the result [measured in quarters] is equal because the quarter of a quarter is half of an eighth.

Example: We wish to multiply 3 fifths by 4 fifths. The denominator is 5. For the 3 fifths we take 3, and for the 4 fifths 4. We multiply 4 by 3 yielding 12, which is the product. [Taken with respect to 5 squared] this gives 2 fifths of the square[8] and 2 fifths of a fifth.

And if we are given fractions of two kinds, and are instructed to multiply 2 thirds of one by 3 of its quarters, we seek the denominator for both, multiplying 3 by 4, which is the denominator 12 whose square is 144. So for 2 thirds we take 8 [out of 12], and for 3 quarters 9 [out of 12].[9] We multiply 8 by 9 and get 72, which is half of 144, the square of the denominator. So the result is half of one.

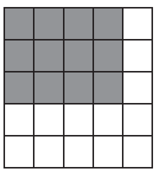
And if you multiply 2 by 3, you also have half the denominator, which is 12. So if you have two [different] denominators it is easier, as there is no need to square the denominator. Consider only the product of one denominator multiplied by the other as if it were the square [in the product of fractions with equal denominators], and divide by it.

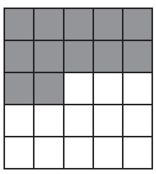
Example: We take one denominator 3 for the said thirds and the other denominator 4 for the said quarters. Multiply the one denominator 3 by the other denominator 4 yielding the required 12, with respect to which we take the [following] product: for the 2 thirds we take two (out of the 3), and from 3 quarters we take three (out of 4). We multiply 2 by 3, yielding 6, which is half the product of the denominators.

Commercial problems solved by proportion rules (from Chapter 6)

The following is an example of the usage of proportion in commercial problems (the Rule of Three). Note the use of the wheel symbol for both zero and the position of the unknown number in the diagrams. The first example requires a double use of proportion (Rule of Five), and the second applies the Rule of Three in a false position (making a guess and then rescaling

[8]Does the "square" here suggest a figurative model—a square divided into 5 rows and 5 columns? With such a figurative model in mind, it is easy to see that the product of 3 rows by 4 columns equals two full rows (fifths) and two extra subsquares (fifths of fifths).





However, the term "square" could simply refer to the squared number, 25. The elliptic formulation would then be odd, but not impossible.

[9]Here Ibn Ezra is finding a common denominator for the two fractions to be multiplied. This might strike us as odd, but may indicate the use of a figurative model as above, where an equal subdivision into 12 rows and 12 columns is preferred over an unequal 3-by-4 subdivision. Alternatively, this preference for a common denominator may be the influence of sexagesimal-like systems, where a multiplication of minutes by minutes (sixtieth parts) is measured in seconds (3600th parts), or may perhaps indicate a general preference for homogeneity in arithmetical operations.

to fit the required result). Note also that the famous "tree" question (which was treated, among others, by Fibonacci) is already referred to as a standard reference problem here.[10]

Question: Reuven hired Simon to carry on his beast of burden 13 measures of wheat over 17 miles for a payment of 19 *pashuṭ*s.[11] He carried seven measures over 11 miles. How much shall he be paid? Do thus. You should apply proportions twice, as there's no other way to get it. Suppose he carried the 7 measures according to terms, which is 17 miles. Draw this diagram.

13 7
19 $\underline{0}$[12]

We multiply the extremes, which are 7 and 19, yielding 133. We divide them by 13, yielding 10 wholes and leaving 3 parts of 13 in one *pashuṭ*. But since he carried but the 7 measures for only 11 miles, we should apply another proportion and another diagram thus: 11, 17, wheel, 10 and 3 parts of 13. Here is the diagram:

17 11
3 10 $\underline{0}$

Since we should multiply the extremes and divide the result by 17, and we have in the fourth 3 parts of 13, we should seek one denominator for both. We find it by multiplying 13 by 17, yielding 221, which is the denominator and is one whole. 3 parts of 13 are 51 of 221. We then multiply 11 by 10, yielding 110, and multiply 11 wholes by 51 parts yielding 561. We divide by 221, which is the whole one, yielding 2 wholes, which we join with 110 to make 112, and a remainder of 119 parts of 221. We divide 112 by 17 wholes yielding 6 whole *pashuṭ*s and a remainder of 10 whole *pashuṭ*s, which we count with respect to 221 [yielding 2,210], and add the remaining 119 parts of one *pashuṭ*, which is 221 parts, yielding 2,329. We divide by 17, yielding 137. The total is 6 whole *pashuṭ*s and 137 parts of which 221 make a whole.

. . .

[Question:] We subtract from an estate its fifth, seventh and ninth, leaving 10. [To find the estate,] subtract 143 which are the fractions [1/5, 1/7 and 1/9] out of 315, which is the denominator [as calculated earlier], leaving 172. We do the proportion thus.

$\underline{0}$ 10
315 172

We multiply 10 by 315 yielding 3,150. We divide by 172, yielding 18 wholes and 54 parts of 172. If we take away the fifth, seventh and ninth of this number, we're left with 10 wholes.

[10] See section II-2-2 on Leonardo of Pisa, *Liber abbaci*, in Chapter 1 of this *Sourcebook*.

[11] Literally, *pashuṭ* means "simple," but here it is used as the denomination of a coin, which is one-twelfth of a *dinar*. In fact, the *pashut* is almost surely the *denier*, a paper-thin lightweight coin of mixed silver and copper, and the standard coin of Europe between the eighth and thirteenth centuries. The *dinar* is just as surely the *gros tournois*, a heavy royal French silver coin introduced by King Louis IX of France in 1266, which became the standard coin of Europe from the late thirteenth century. For more information, see [Simonson, 2000b].

[12] The manuscripts use the same symbol for zero and the unknown quantity in the Rule of Three. We underline the latter to prevent confusion.

And the same method applies to the question of the tree, which has a third in the water and a quarter in the ground, leaving 10 cubits above water. How tall is the entire tree? We seek a number that has a third and a quarter, which is 12. Its third and quarter joined are 7. We subtract them from 12, leaving 5. We do the proportion thus.

<u>0</u> 10
12 5

We multiply the extremes, yielding 120. We divide by 5, yielding 24. This is the height of the entire tree, because its third is eight, and its quarter six, which joined together make 14, and when we subtract them from 24, there remain 10 wholes, no less and no more.

One of the motivations for scholarly presentations of algebra and proto-algebra in the Islamic world was the concern with sharing inheritances according to Muslim law. Here we see this concern reflected from the point of view of Jewish law as well.

Question: Jacob died. His son, Reuven produced a deed signed by two valid witnesses that his father Jacob gave him alone his entire estate upon his death. His son Simon also produced a deed that his father ordered to give him half of his estate upon his death. Levi too produced a deed that his father ordered to give him a third of his estate. And Judah also produced a deed that he be given a quarter of his estate. And all carry the same day, time and hour in Jerusalem where time is reckoned.

The scholars of Israel divide it by the claim of each, and the gentile scholars by the proportion of the estate of each. The scholars of arithmetic reckon the estate as one, and when you add its half and third and quarter, the total is two and half a sixth. Consider the one whole as sixty, which has all the mentioned parts, yielding a total of 125. Or we can consider the one whole as 12 and the fractions 13; the final result will be the same either way. Let us calculate Reuven's part according to the proportion of his estate. We will calculate the proportion with respect to sixty, as he requests the entire estate. Say that the estate is 10 dinars which are 120 *pashuṭ*s. This is the proportion rule for Reuven's estate.

<u>0</u> 60
120 125

We multiply the extremes, yielding 7,200, and divide by 125, yielding 57 *pashuṭ*s and 75 parts, which is Reuven's part. [Using this same method, Simon gets 28 *pashuṭ*s and 100 parts, Levi gets 19 *pashuṭ*s and 25 parts, and Judah gets 14 *pashuṭ*s and 50 parts.]. . . .

According to the scholars of Israel, the three elder brothers say to Judah their brother: you contest only 30 *pashuṭ*s [a quarter of the estate], and all our claims to them are equal. Take 7 and a half, which is a quarter, and leave us. And each of the three brothers will also take as much. Then Reuven says to Levi: you only contest 40 *pashuṭ*s, and have already taken your share of the 30 which all four of us contested. Take a third of the 10, which is [the remaining] quarter of 40, and leave us. And so Levi's share is ten and five sixths. . . . Reuven then says to Simon: you only contest half the estate, which is 60, and the other half is all mine. You've already taken your share of the 40, so you and I contest only 20. Take half of that and leave me. So Simon's share is twenty and five sixths *pashuṭ*, and Reuven's part is eighty and five sixths of one *pashuṭ*.[13]

[13]The method of the "scholars of Israel" is reminiscent of Pascal's argument in his letter to Fermat concerning the equitable division of stakes in a game interrupted before its conclusion. See [David, 1998, pp. 85–88.]

Extracting square roots (from Chapter 7)

Here we see the technique for extracting square roots. Ibn Ezra first shows how to derive the integer part of the root. Then, after some motivational discussion concerning identities involving squares, he uses rescaling to obtain sexagesimal approximations of irrational roots.

Note that there are 3 squares in the first rank [one-digit numbers], namely 1, 4 and 9. In the second rank [two-digit numbers] there are 6 [squares], namely 16, 25, 36, 49, 64 and 81. And all the ranks that follow these two proceed in the same way: all odd ranks take after the first rank, and all even ranks take after the second rank. Indeed, squares that take after squares in the first rank [namely, one digit squares multiplied by an even power of 10] always have one digit, and those in the second rank have two digits, as do all the numbers that take after them [namely, those multiplied by an even power of 10]. From this analogy you can tell the squares that precede or follow them.

If you know the root of a number in the first or second rank, and want to know the rank of the root of the analogous number [multiplied by a power of 10], do as follows. Know that [the root of] what lies in the first rank is units, in the third rank tens, in the fifth—hundreds, in the seventh—thousands, in the ninth—ten thousands, in the eleventh—hundred thousands, and so on without end skipping from one odd number to the next odd number. And the units are the roots of the second rank, of the fourth rank—tens, of the sixth—hundreds, of the eighth—thousands, of the tenth—ten thousands, of the twelfth—hundred thousands, ever skipping from even to even.[14]

And now I will instruct you [on] what to do when you know the analogous square and its root. Subtract the square from the desired number [whose root you wish to find], but make sure you take only the square preceding your number. Take the distance between your number and the square, and divide by twice the root of the preceding square. Now don't subtract from your number as much as you can [namely the result of the division times twice the root]—leave enough to fit the square of the result of the division. If the distance to the preceding square equals what comes from the division multiplied by twice the root, joined together with the square of what comes from the division, then the square is true. . . .[15]

Example: We wish to know the square preceding two hundred. This number is of the third rank, which is odd, so we consider the first rank, where the squares are 1, 4 and 9, and their analogous squares are a hundred, 4 hundred and 9 hundred. A hundred is the preceding square, and its root is 10, as we said: the roots which are units for the first rank are tens for the third. Subtract the square [of the root, 100] from our number [200], leaving 100. We already said that the root is 10, so its double is 20. If we divide 100 by 20,

[14] Ibn Ezra explains that the integer part of the root of a decimal number of the form $x\ \overbrace{00\ldots0}^{2n}$, where x is a square integer with one or two digits, is of the form $y\ \overbrace{00\ldots0}^{n}$, where y is a single-digit integer.

[15] The algorithm is as follows. To extract the root of x, take a, an integer approximating the root from below (following the instructions of the previous paragraph). Then consider $x' = x - a^2$. To obtain the next digit of the root, a similar subtraction should be applied to the remainder x'. However, now we should remove from x' not only the square of the chosen integer, b, but also the term $2ab$. So b may be obtained from the quotient $x'/2a$, except that this quotient must be reduced so that b^2 can fit into $x' - 2ab$. The process is reiterated until the integer part of the root of x is obtained.

and give the result 5, we are left with nothing from which to take the square of the result of the division [after subtracting 5 times 20 from the remaining 100]. So we give 4, which multiplied by 20 makes 80, leaving 20. We then subtract 16 which is the square of what comes from the division, leaving 4. Subtract it from the two hundred, leaving 196, which is the square preceding two hundred. Add 4, which comes from the division, to the first root which was 10. This makes 14, which is the true square root [of 196].

The above calculations obviously relate to what can anachronistically be expressed as $\frac{\sqrt{100a}}{10} = \sqrt{a}$ and $(a+b)^2 - a^2 = 2ab + b^2$. These two identities are easy to understand in hindsight, but it is not clear how they would lead to the root extraction algorithm in a pre-algebraic, intuitive way. The seemingly unremarkable examples in the following paragraph show how a practice of squaring numbers and retracing one's steps can produce such practical intuition. In the first example, sexagesimal-like fractions (here actually septagesimal ones) bring up the rescaling phenomena related to the first identity above, while the second example shows how the intuitive division of a number into an integer and a fractional part makes salient the structure of the second identity.

Example for a calculation that cannot be divided by 60. We multiply 4 sevenths by 4 sevenths. According to the scholars of arithmetic, we multiply 4 by 4 to get 16, divide by 7 [twice] and get 2 sevenths and 2 sevenths of a seventh. According to a way similar to that of the scholars of astrology, make 70 parts, so 4 sevenths are 40 [minute-like parts]. We multiply 40 by 40 to get 1600, divide by 70, and get 22 minutes and 60 seconds, which is the square. Now if he who posed the question turns it around, and asks for the root of this square, so shall we turn the 22 minutes into seconds, and add the 60 seconds that we had in order to get 1600. We consider them as wholes, whose root is 40, and then consider them as minutes, and this is indeed the root.

. . .

Another example: It is said that the square is 11 and a ninth. What is the root? Since it is said to have a ninth, this ninth indicates that there's a third in the root of this square. Subtract the ninth, which is the square [of 1/3] or fraction of fraction, and 11 wholes remain. Their distance from the previous [whole] square [which is 9] is 2 wholes. We turn them into minutes, yielding 120. We divide by twice the preceding root, which is 6, and get 20. So the root is 3 wholes and 20 parts, [which are] a third.

The next paragraph shows how to derive a better approximation of the root of 2 by taking the root of 20,000. This calculation comes after the root of 2 was calculated via the root of 200, yielding the result 1 $24'$ $52''$.

Let's extract this root [of 2] again from 20,000. This number is analogous to units, and 10,000 is analogous to 1, so this is the analogous square. We subtract 10,000 from our number and 10,000 remain. We divide by twice the root, which is 200, but we don't give it as much as we can, leaving enough for the square of the result of the division. So we give it 40, and [after subtracting 40 times 200], two thousands remain. We subtract 1,600, which is the square of the result of the division, leaving 400. Our [cumulative approximation for the] root is 140, and its double is 280. We divide the remainder by it, give it one, and 120 remain. We subtract 1, which is the square of 1, leaving 119, and our [cumulative approximation for the] root is 141. We turn the remainder into minutes, yielding 7,140, and divide by 282, which is double our root, yielding 25 minutes and 19 seconds. [To get the square root of 2], we divide the total of wholes and parts by 100, yielding 1 whole 24′ 51″ 11‴, and this is more precise than the former calculation [derived from the root of 200, which gave 1;24,52].

Mathematics as a natural science (from Chapter 5)

The following statement concludes a short discussion of the evaluation of pi. It likens the discovery of such evaluations to an empirical process rather than a deductive one.

And likewise [i.e., as in the evaluation of pi,] in natural history the scholars found by way of trial and error the true properties of herbs and stones and the parts of the human body, and none of them knows why it is so, except the blessed unfathomable God.

2. AARON BEN ISAAC, *ARITHMETIC*

This section was prepared by Naomi Aradi

The only information known to us today about Aaron and his arithmetical work comes from a single surviving manuscript in Turin. A brief bibliographical reference concerning the manuscript is available in [Steinschneider, 1906, p. 197]. Since the upper part of this manuscript was apparently burned, the first few lines of every folio are illegible. The disruptions in the text indicate that it may be an autograph. The manuscript contains Aaron's arithmetical textbook only. The bibliographers estimate that the manuscript is from fifteenth-century Spain.

The work is divided into three sections. The first section deals with arithmetical operations (addition, multiplication, subtraction, and division) and has four parts: the first three consider respectively operations with integers alone, with fractions alone, and with both integers and fractions, while the fourth part considers roots and progressions.

The second section is devoted to word problems arranged in five parts. The first four parts present addition, multiplication, subtraction, and division problems, respectively, each part in turn discussing cases involving integers alone, fractions alone, and a mixture of both types in an orderly manner. The fifth part concerns the double false position.

The third section is devoted to ratios, including issues related to number theory, such as the discussion on amicable numbers examined in section II-7 below.

Aaron mentions Abraham ibn Ezra when he attributes to him one of his word problems [f. 162a]. Yet this problem does not appear in Ibn Ezra's famous *Book of Number*, which is not mentioned by name in Aaron's work. As he discusses ratios, Aaron notes Greek mathematical terms (in Hebrew letters). This raises the possibility that he knew Greek and perhaps even relied directly on Greek texts, such as the *Introduction to Arithmetic* by Nicomachus.

Included here are some excerpts from the preface, describing Aaron's biographical background, and a mathematical-philosophical discussion of numbers and decimal representations.

I, the aforementioned Aaron, from the day I began to engage in the craft of weaving gold and silk images, although proficient in it with accuracy and precision, realized that I lack some of the arithmetical methods required for the subtlety of the craft. Though I had some knowledge of it, I put my mind and energy to investigate the general practices of number and some of its particular properties, as human knowledge cannot comprehend them all.

So I saw fit to write what came within my reach, and put it together into a composition containing the rules of number and most of its many properties, so as not to forget with the passing of events and time what I had learned and to teach them to my sons, God willing.

If a scholar gets hold of this treatise or of *Miqnat Kesef* [acquired property][16] or *ʿIbbur* [intercalation] or . . . *Ṣurat Haʿolam* [the shape of the world], which is *Alfarghani* in another form, and finds that my language is not that of a learned man, he should not judge me harshly, but give me the benefit of the doubt, because I am a craftsman, as I said, and composed these treatises for my son Joseph. Now I add that, by my sins, he is dead, and these compositions are left as they stood, without order and unedited. Indeed, he was to put them in order and edit them, but now he lies deep, and I am at the end of my days, and I never got to teach him. And so, they are as naught.

I say that the kinds of number are two. The first is the number deprived of substance, speech or thought, which is unlimited, because it is a number *in potentia* rather than *in actu*, and has no end. The second kind, which this treatise concerns, is the number initially delimited and bounded by thought. Then, through the rational capacity, one can pronounce it or write it, as one wishes. This number is thus counted and limited. It is divided into two: even and odd, whose foundation is one, because the counted number is a sum of units.

. . .

A necessary comment on the knowledge of letters which are used in arithmetic: I say that since the number deprived of substance is unlimited, the arithmeticians had to use endless figures or letters.

[A few corrupt lines on the finiteness of letters in all languages follow.]

Since there is no letter after ת [the last letter of the Hebrew alphabet, designating 400], you cannot find a letter to express a greater number. Even if you use the letters which indicate large numbers, they will not be sufficient due to the greatness of that number, and if you repeat them several times in writing [using them additively to express higher numbers], much confusion will ensue.

The uncounted number is without end, and the letters have an end, and that which has an end cannot count the endless. In other words, the smaller cannot count the larger. So the arithmeticians agreed to set 9 figures with endless ranks. The nine figures are like substance, and in the ranks they are of different form. The first, for example, is one, and can be ten or a hundred or a thousand or larger numbers without end. The same goes for the other 9 figures set by some scholar. These are the figures accompanied by the corresponding Hebrew letters, which you may use to deal with numbers, as I do in this treatise.

א ב ג ד ה ו ז ח ט 0

0 9 8 7 6 5 4 3 2 1

[16]This term is quoted from Exodus 12:44, where it refers to purchased slaves.

So the ranks are due to the largeness of numbers, and the sparsity of letters is due to the ranks.

3. IMMANUEL BEN JACOB BONFILS, *ON DECIMAL NUMBERS AND FRACTIONS*

Immanuel lived in Orange and Tarascon in Provence (ca. 1300–1377). He made his name as an astronomer, having composed several astronomical treatises and tables, the most famous of which, *Six Wings*, was subsequently translated into Latin and Greek [Solon, 1970]. He also translated from Latin into Hebrew. His mathematical work focuses on the calculation of roots and on circle measurements. His decimal treatment of fractions is the earliest surviving treatment to be recorded in Europe.

This short note on decimals was first identified in a small astronomical codex. Parts of it are available in several manuscripts, but only three of these manuscripts include the decimal treatment of fractions. The note treats multiplication and division of decimal fractions, division of sexagesimal fractions, and rescaling numbers in decimal and sexagesimal systems for the purpose of root extraction. The treatment of decimal fractions and integers is completely homogenized: each decimal place, either integer or fractional, is assigned a positive degree, and these degrees are used to calculate the decimal place of the product or quotient. The exception is units, which are not treated as "zero degree" integers but as a special case. The selection below includes a short fragment on sexagesimal division, which places the decimal treatment in context.

Know that the unit is divided into ten parts which are called prime fractions, and each prime is divided into ten parts which are called seconds, and so on without end. I also want to call to your attention that I am calling the degrees of the tens first wholes, and the hundreds second wholes, and so on without end. The degree of the units, however, I am calling by their name, units, for it is an intermediate between the wholes and the fractions. Therefore, if one multiplies units with units the result is units [which is not the case for any other degree].[17]

Furthermore, I am calling the degrees whose name is greater "greater in name." I mean by that: I am calling the thirds greater in name than the seconds, for the thirds are derived from three, while the seconds from two. Similarly, the fourths are greater in name than the thirds and the fifths than the fourths. This applies to the wholes as well as to the fractions. Furthermore, when I say: add this name to that name, or subtract this name from that name, I mean by that: add the name of the seconds to the name of the thirds, yielding fifths, or the name of the seconds to the name of the seconds, yielding fourths. Or also, subtract the name of the seconds from the name of the thirds, leaving firsts, or the name of the seconds from the name of the seconds, with nothing left, so it falls into the degree of the units. This applies to the wholes as well as to the fractions. If you subtract a large name from a small name, as when we say: let us subtract the name of fourths from the name of seconds, be it among wholes or among fractions, then it will come into the degree of the seconds on the other side. For instance, when we say: let us subtract the name of the fourths in the fractions from the name of the seconds also in the fractions, then it falls into the degree of the second wholes. Similarly, if we say the same

[17] Words in curly brackets come from a new manuscript recently reconstructed by Naomi Aradi.

with regard to the wholes, i.e., if we want to subtract the name of the fourth wholes from the name of the second wholes, then it falls into the second fractions.

If you multiply a number by a number, both being wholes or both fractions, add the name of the ranks, and there the product will be among the wholes if both are wholes, and among the fractions if both are fractions. If, however, the one is a whole and the other a fraction, if they have the same name, then the product will fall into the degree of the units. But if the name of one is greater than the other, then subtract the smaller from the greater, and as the number of the remaining name, there the product falls—among the wholes if the name of the wholes is greater, or among the fractions if the name of the fractions is greater. [As when you multiply 3 second wholes by 2 seventh fractions, you subtract 2 from 7, leaving 5. The multiplication yields 6 fifth fractions. For 3 second fractions by 2 seventh wholes, the product will be 6 fifth wholes. Here's a diagram for that.][18]

	Hundreds of thousands	*Tens of thousands*	*Thousands*	*Hundreds*	*Tens*	*Units*	*Primes*	*Seconds*	*Thirds*	*Fourths*	*Fifths*	*Sixths*
			2* 3	5 1	4 2*	1 5	3 4	2 3	1 2			
{1} [6]	[1] 5 2	2 6 5	3 5 2 0	9 1 0 7 8	6 3 2 1 1 6	3 2 7 5 7 6 4	1 5 6 4 2 0	2 0 2 3 8	5 8 9 2	4 6 6	3 4	2
7	9	4	2*	7	2	5	9	7	5*	6	7	2

If you divide a number by a number, both wholes or both fractions, and the name of their ranks is the same, then the quotient will fall into the degree of the units. For if you subtract the one name from the other name, nothing remains, and it falls into the degree of units. If the dividend is greater in name than the divisor, subtract the name of the divisor from the name of the dividend, and as the number of the name of the remainder, there the quotient will fall on that side, i.e., among the wholes if they were whole, or among the fractions if they were fractions. If, however, the divisor is greater [in name] than the dividend, then subtract the name of the dividend from the name of the divisor, and as the number of the name of the remainder, there the quotient will fall on the opposite side, i.e.,

[18] The example in the manuscript is 4541.321 times 3135.432. The numbers are originally presented as letters. Naomi Aradi noted that the calculation diagram in the manuscript would make most sense if the first digit of the first multiplicand is rendered as 2 instead of 4 and the third digit of the second multiplicand as 2 instead of 3. Then only a couple of digits at the bottom line need to be amended. The differences between the original and revised versions are marked with asterisks.

among the fractions if both were wholes, or among the wholes if both were fractions. If, however, the one is a whole and the second is a fraction, the name of their ranks being the same or different, then add the name of the ranks, and as the number of the name of the result, there the quotient will fall—among the fractions if the dividend is a fraction, or among the wholes if the dividend is a whole.

. . .

If you want to divide a number of [sexagesimal] degrees, minutes and seconds by another smaller or greater number, then take the number of the lower row, which is the number by which the number of the upper row is divided, and break it all into the kind of the smallest fraction in it. For example, if its smallest fraction is seconds then reduce it all into seconds, and if thirds then reduce it all into thirds, and so on in the same manner. Thereupon take the number of the upper row, which is the number divided by the number of the lower row, and break it into such a kind of fractions, that its distance from the kind of fractions into which you have broken the lower be equal to the distance of the kind of fractions which you want to obtain in the quotient from the degrees.

For example: if you have broken the number of the lower row into the kind of the seconds and you want to obtain in the quotient thirds (or any other kind that you want, but let us suppose in this example that you want thirds), then break the number of the upper row into the kind of fifths, which is removed from the seconds by three degrees, like the distance of the thirds from the end of the degrees, and then you will obtain thirds in the quotient. And if you want to be more accurate and obtain fourths in the quotient, break the number of the upper row into sixths, which is removed from the seconds by four degrees, like the distance of the fourths from the end of the degrees, and then you will obtain fourths in the quotient.

4. JACOB CANPANṬON, *BAR NOTEN ṬAʾAM*

This section was prepared by Naomi Aradi

Rabbi Jacob Canpanṭon lived in Castile in the fourteenth and fifteenth centuries. In Hebrew historiography, he is said to be "one of the rabbis of Spain" and a student of Rabbi Ḥasdai Crescas, who wrote the famous philosophical work *ʾor Hashem*, and the father of Isaac Canpanṭon (1360–1463), *Gaʾon* of Castile. In addition, it is stated that he wrote books on arithmetic, astronomy and the Torah [Hacohen, 1967–1970, vol. 5, p. 94]. According to [Steinschneider, 1893–1901, p. 186], Jacob apparently was already a teacher in 1406 and was no longer alive in 1439. He worked as a mediator in the field of medicine while preparing a Hebrew summary of the Arabic commentary by Solomon ibn Jaīsh on the *Canon* of Avicenna.

Canpanṭon's arithmetical textbook survives in the single manuscript located at the British Library in London. It is titled *Bar Noten Ṭaʾam*, based on a Talmudic saying, indicating the author's intention to explain the reasoning (*ṭeʾamim*, literally, reasons) behind the arithmetic operations, and not merely describe the procedures. The treatise begins with an introduction phrased as a long rhymed poem embedded with biblical and rabbinic expressions describing the circumstances that led the author to write the composition. The treatise was written at the request of one Rabbi Joel ben Daʾud, a close friend of Canpanṭon, who wished to learn mathematics. The book is divided into two sections. The first section is devoted to integers and contains six chapters on addition, subtraction, multiplication, division, proportions, and roots. The second section deals with fractions and discusses the following issues: conversion, fractions of fractions, equalization, addition, subtraction, multiplication, division, ratios, roots, common denominators, completion, and shortcuts.

The passages below are taken from the sixth chapter of the first section dedicated to extraction of roots of integers. In this chapter, Canpanṭon describes iterative root extraction algorithms in his typical lucid and elaborate manner, including a detailed comparative error analysis. As far as is known, the formula $\sqrt{a^2+r} = a + \frac{r}{2a+1}$ was used in Eastern Arab mathematics, but not in the West [Harbili, 2011]. Canpanṭon analyzed when it outperforms the Western formula $\sqrt{a^2+r} = a + \frac{r}{2a}$. Canpanṭon also suggests and iterates the formula $\sqrt{a^2+r} = a + \frac{r}{2a+\frac{r}{2a}}$. I am not aware of earlier precedents for these two contributions.

The discussion starts with a description of the standard algorithm for extracting the integer closest to the square root of a given integer. It then considers further approximations for the roots of non-square integers, where the extracted integer root leaves a remainder.

When there is a remainder after you have completed the extraction of the [integer part of the] root, and you wish to come closer to the truth, consider this remainder. If it is less than the [integer part of the] root, double the root and set it as a denominator to divide the remainder. The result is the addition to the integer [part of the root obtained by the algorithm] in the [new approximate] root. If the remainder is greater or equal to the [integer part of the] root, and you do not intend to come closer to the root except by this step alone, then double the root, add one, and divide the remainder by the sum. The result is the fractions added in the [new approximate] root to the initial integer [obtained by the algorithm].[19]

If you wish to come closer to the truth, even if the truth is invisible to all living beings, as Euclid proved, multiply the integer and fractions by themselves ... and the result will exceed or fall short of the initial number [whose root is being extracted]. Double the root, as we said, and divide the excess or deficit by the result. Subtract the result from the preceding fractions if the number [whose root is being extracted] is less than the square of the root that you extracted in the previous stage; and if the square is less than the number [whose root is being extracted], add the result to the preceding fractions. The sum or the remainder will be the fraction added in the root to the initial integer [part of the root obtained by the algorithm].[20]

Here Canpanṭon presents the following example: on extracting the root of 10,375, the result is 101 and the remainder is 174. Since 174 is larger than 101, the new root is 101 and 174/203, which equals 101 and 6/7. Then, since the square of this number is less than 10,375 by $\frac{6}{7} \times \frac{1}{7}$, the next approximation would add $\left(\frac{6}{7} \times \frac{1}{7}\right)/\left(2 \times \left(101\frac{6}{7}\right)\right)$.

You come ever closer to the truth, but you will never attain it. If you look closely, you will see that you can get the [same approximation] with less effort. Indeed, consider the

[19]In anachronistic terms, Canpanṭon suggests that when extracting the root of $a^2 + r$, where a and r are integers and $r < a$, then the first approximation for the root should be $a + \frac{r}{2a}$, and if $r \geq a$, then the approximate root should be $a + \frac{r}{2a+1}$.

[20]Canpanṭon states the following general principle for iterative root approximation: if a (not necessarily an integer) approximates the root of $a^2 + r$, then the next approximation should be $a + \frac{r}{2a}$, and the same goes for a subtracted r.

fraction attained. If it is an addition [to the integer part of the root], and it was produced by adding 1 to twice the root [in the denominator], then find the product of this fraction by its complement [with respect to 1]. This product will be the deficit of the square [of the approximate root] with respect to the initial number [whose root is being extracted],[21] and should therefore be divided by double the [approximate] root itself and added to this [approximate] root. This is clearly visible in the previous example (which had [a fraction] added and involved adding one [in the denominator]), where the fraction was six sevenths. Its complement with respect to one is one seventh, and their product is six sevenths of a seventh; this is indeed the deficit we found with respect to the original number [whose root was extracted], and so we instructed to divide it by double the [approximate] root and add [the result] to this root.

But if the additional [fraction] did not involve adding 1 [in the denominator] and falling short [of the initial number whose root is being extracted], then multiply the [additional] fraction by itself, and divide by double the [approximate] root, because this is the excess of the square of the [approximate] root over the original number [whose root is being extracted].[22] We subtract the result from the previous [approximate] root, and so on.

[Do] this if you wish to approach the truth by repeating the procedure, because the more you repeat, the nearer you come to the truth, even if you can never attain it, as we have explained. [If you repeat the procedure], never add 1 to double the root, even if the remainder is very large with respect to the [approximate] root, so as to avoid confusion, for [adding 1] was instructed only for a single [approximation] step. Adding 1 when the remainder is greater or equal to the [approximate] root improves the approximation, as I explained, but if one repeats the procedure, one does not need this addition, because by repeating the procedure one approaches [the truth] very closely even without adding 1. It is better not to add it, so as to maintain a standard form of procedure and prevent confusion.

The reason we say that if we have a remainder smaller than the [approximate] root, then we should divide it by double the root is the following. That which we add to the root will add to the square its product by twice the previous root and its product by itself, as we explained with regard to integers. But we proceed as if it only adds its product by twice the root. If this were true . . . then we would have this product, which equals the excess of the number [whose root is being extracted] over the square of the integer [received through the root extraction algorithm], and . . . we would reach the required result. . . . [But] that which is added to the root further adds to the square its product by itself. . . . Therefore, when we multiply the root with the addition by itself, the square will exceed the initial number [whose root is being extracted] by the square of the addition.[23]

[21] Canpanṭon claims that when the root of $a^2 + r$ is approximated by $a + \frac{r}{2a+1}$, then the error (the original number less the squared approximation) is $\frac{r}{2a+1} \times \left(1 - \frac{r}{2a+1}\right) = \frac{r}{2a+1} \times \frac{2a+1-r}{2a+1}$.

[22] Here it is stated that when the root of $a^2 + r$ is approximated by $a + \frac{r}{2a}$, then the error (the squared approximation less the original number) is $\left(\frac{r}{2a}\right)^2$. In the next paragraph Canpanṭon explains that for a reiterated approximation, one should always use this, rather than the previous approximation.

[23] To justify the above error estimate, Canpanṭon uses the equality, which in anachronistic terms would be rendered $\left(a + \frac{r}{2a}\right)^2 = a^2 + 2a \times \frac{r}{2a} + \left(\frac{r}{2a}\right)^2$.

The explanation is then repeated to obtain the next (subtractive) step of the approximation and its error term, which in anachronistic terms reads: $a + \frac{r}{2a} - \left(\frac{r}{2a}\right)^2 / \left(2\left(a + \frac{r}{2a}\right)\right)$ with the error $\left(\left(\frac{r}{2a}\right)^2 / \left(2\left(a + \frac{r}{2a}\right)\right)\right)^2$.

So, when we do not add 1 [to the denominator], and wish to approach the truth, [in the first step] we should only add the fraction of the first step [i.e., the remainder divided by twice the integer approximating the root]. But from there on we must divide the square of the fraction produced at that step by twice the previous [approximate] root. And the result will ever be subtracted from the previous [approximate] root.

Next comes a detailed demonstration of the above reasoning in the extraction of the root of 7. The integer received through the extraction algorithm is 2, and the remainder is 3. The first approximation involving a fraction is therefore $2\frac{3}{4}$. The square of this exceeds 7 by $\left(\frac{3}{4}\right)^2 = \frac{9}{16} = \frac{2}{4} + \frac{1}{4} \times \frac{1}{4}$. Therefore, the next approximation is $2\frac{3}{4} - \frac{\frac{2}{4}+\frac{1}{4}\times\frac{1}{4}}{2\times(2\frac{3}{4})} = 2\frac{3}{4} - \frac{9}{11} \times \frac{1}{2} \times \frac{1}{4} = 2\frac{2}{4} + \frac{1}{2} \times \frac{1}{4} + \frac{2}{11} \times \frac{1}{2} \times \frac{1}{4}$. The square of this then falls short of 7 by $\left(\frac{9}{11} \times \frac{1}{2} \times \frac{1}{4}\right)^2$, and so on.

The reason we say that when the remainder is greater than or equal to the [integer part of the] root, we should divide it by double the root plus one (as long as we do not intend to repeat the procedure so as to further approach the truth and restrict ourselves to this step only), is that if we had not added one, the square of the root consisting of the integer and fraction would exceed the number [whose root is being extracted] by the square of the fraction received in the division. But this would be a quarter or more. For if [the remainder] is equal to the root itself, and we divide it by double the root, the result of the division will be a half. Its square (namely, its product by itself, which is the excess) will then be an entire quarter. And if the remainder is greater than the root, when we divide it by double the root the result will be more than a half, and its square more than a quarter.

Canpanṭon gives the following example: without adding 1, the root of 6 is approximated by $2\frac{1}{2}$, whose square exceeds 6 by $\frac{1}{4}$; if we add 1 to the denominator, we get $2\frac{2}{5}$, whose square is less than 6 by $\frac{2}{5} \times \frac{3}{5}$, which is smaller than $\frac{1}{4}$.

If you divide [the remainder] by double the root plus 1, the square of the root will be less than the sought number by the product of the quotient and its complement with respect to one, which can never in any way reach a quarter. For the product of a portion of a line or a number by its complement never reaches a quarter. Because if you multiply its half by its half the result will be a quarter, and clearly, if you multiply its smaller portion by its larger [complementary] portion, the product will not be a quarter but smaller than [a quarter] by the square of their distances from half the line or the number.[24]

Canpanṭon then gives examples for the above: $\frac{1}{4} \times \frac{3}{4}$ is smaller than $\frac{1}{2} \times \frac{1}{2}$ by $\frac{1}{4} \times \frac{1}{4}$; 5×7 is smaller than 6×6 by 1×1; 9×3 is smaller than 6×6 by 3×3. In the context of root extraction, the following examples are given: the approximate root of 7 is $2\frac{3}{5}$, and its square is less than 7 by $\frac{3}{5} \times \frac{2}{5} = \frac{1}{5} + \frac{1}{5} \times \frac{1}{5}$; the approximate root of 6 is $2\frac{2}{5}$, and its square is less than 6 by $\frac{2}{5} \times \frac{3}{5} = \frac{1}{5} + \frac{1}{5} \times \frac{1}{5}$.

[24]Canpanṭon shows that when r is larger than or equal to a, the error term of the approximation $a + \frac{r}{2a}$ exceeds $\frac{1}{4}$, whereas the error terms of the approximation $a + \frac{r}{2a+1}$ will be smaller than $\frac{1}{4}$. The reasoning goes through a geometric interpretation, and an implicit reference to *Elements* II.5.

> The reason is that the remainder equals the result of division multiplied by twice the [previous approximate] root plus one. . . . The addition of the result to the previous [approximate] root, however, will add to the square its product by twice the previous root and its product by itself. But its product by itself subtracted from its product by 1 is its product by its complement [with respect to 1].[25] (For example, the product of $\frac{1}{3}$ by 1 equals its product by the parts [of 1], namely by $\frac{1}{3}$, which is itself, and by $\frac{2}{3}$, which is its complement with respect to one; this is clear).

Canpanṭon supplies an example for the reason we do not add one when the remainder is small: if we approximate the root of 29 by $5\frac{4}{10}$, the error is $\frac{16}{100} = \frac{1}{4} - \frac{9}{100}$; if we approximate it by $5\frac{4}{11}$, the error will be $\frac{4}{11} \times \frac{7}{11} = \frac{1}{4} - \frac{2\frac{1}{4}}{11^2}$. The latter is clearly larger than the former. Then Canpanṭon repeats the instruction not to use the $a + \frac{r}{2a+1}$ approximation when conducting a reiterated approximation. Finally, he suggests a different iterable approximation procedure with its own error analysis. This last method is given without justification.

> If you wish to approach the truth with little effort, add the remainder [i.e., the difference between the given number and the integer part of the root] to the square of double the [integer part of the] root at hand, and divide by it the product of the remainder and double the [integer part of the] root. Add the result to the [integer part of the] root at hand, and this root will be very near the truth. If you wish to approach the truth further, [divide] the cube of the above remainder by the denominator squared.[26]

Canpanṭon concludes with the calculation of the root of 3 according to this formula. The integer approximating the root is 1, and the error term is 2. The approximation formula yields $1 + \frac{2\times1\times2}{(2\times1)^2+2} = 1\frac{2}{3}$. The error term is indeed $\frac{2^3}{\left((2\times1)^2+2\right)^2} = \frac{2}{9}$. Then the procedure is reiterated with $1\frac{2}{3}$ as the root and $\frac{2}{9}$ as the error. The new approximation is $1\frac{112}{153}$, and the resulting error term is $\frac{2}{153^2}$.

5. ELIJAH MIZRAḤI, *SEFER HAMISPAR* (*THE BOOK OF NUMBER*)

This section was prepared by Stela Segev

Elijah Mizraḥi (ca. 1450–1526), also known by the acronym *ha-reʾem*, was born in Constantinople to a family from the Byzantine Empire, rather than from Spanish exiles. At the time, and especially after the expulsion of the Jews from Spain in 1492, the Jewish community of Constantinople was one of the largest and most important Jewish communities in the world. Mizraḥi became a prominent personality in the community, holding many public positions [Hacker, 2007; Ovadia, 1939].

[25] Here Canpanṭon provides a proof of the error term formula $\frac{r}{2a+1} \times \frac{2a+1-r}{2a+1}$ for the approximation $a + \frac{r}{2a+1}$. An anachronistic reconstruction of the argument alluded to here would be: $\left(a + \frac{r}{2a+1}\right)^2 = a^2 + 2a \times \frac{r}{2a+1} + \frac{r}{2a+1} \times \frac{r}{2a+1} = a^2 + 2a \times \frac{r}{2a+1} + 1 \times \frac{r}{2a+1} - \left(1 - \frac{r}{2a+1}\right) \times \frac{r}{2a+1} = a^2 + (2a+1) \times \frac{r}{2a+1} - \left(1 - \frac{r}{2a+1}\right) \times \frac{r}{2a+1} = a^2 + r - \left(1 - \frac{r}{2a+1}\right) \times \frac{r}{2a+1}$.

[26] Canpanṭon suggests the approximation $a + \frac{2ar}{(2a)^2+r}$. This may have been obtained from a sort of interpolation between the underestimate a and the overestimate $a + \frac{r}{2a}$ in the form $a + \frac{r}{a+\left(a+\frac{r}{2a}\right)}$. The error term is calculated as $\frac{r^3}{\left((2a)^2+r\right)^2}$.

Mizraḥi was considered by his contemporaries and by later generations to be the most important rabbinical authority in Constantinople and in the whole Ottoman Empire. He wrote several treatises on religious subjects (e.g., a supercommentary on Rashi's commentary on the Torah), and he was also interested in scientific subjects. Besides *The Book of Number*, he wrote a commentary on Ptolemy's *Almagest*.[27]

Mizraḥi's teacher in secular subjects was Mordechai Comtino, whose treatise *On Reckoning and Measurement* was one of the sources for *The Book of Number*. The Karaite Calev Afendopolo, a colleague, was the author of a commentary on the Hebrew translation of the *Arithmetic* of Nicomachus made from Arabic by Qalonymos ben Qalonymos in 1317.

Mizraḥi's treatise, *The Book of Number* (also known as *Melekhet Hamispar* (*The Number's Craft*)), was widely used during his time as well as later. It can be read in seven extant manuscripts (some of them complete) as well as in the first print edition issued in Constantinople by his son Israel after Mizraḥi's death. The abridged version of the book was partially translated into Latin [Münster, 1546], to be reprinted in 1809.[28]

This essay of approximately 200 pages consists of three articles dealing with arithmetic operations on integer numbers, simple fractions, and sexagesimal fractions. It also includes a chapter on square and cubic roots and on proportions, and a chapter of 99 arithmetic and geometric problems.

For each of his subjects, Mizraḥi first defines and explains the arithmetic operations and related notions, and then describes methods of solution accompanied by examples. He accords special attention to verification techniques. In a separate section at the end of the relevant chapter, Mizraḥi explains and proves his methods. This structure of the book certainly facilitated the publication of the abridged sixteenth-century Latin edition, which contained only the algorithms presented in the first two sections.

Mizraḥi based his work on many sources. He quoted Ibn Ezra, Euclid's *Elements*, and Nicomachus's *Arithmetic*, but it appears that he read also other books, mostly in Hebrew but possibly in other languages (Greek or Arabic). Evidently, he used Comtino's treatise *On Reckoning and Measurement* and also another arithmetic essay by Isaac ben Moshe ʿAli titled *The Art of Number* [Segev, 2010].

Calculating squares by the "thirds method" (from Article I, part 1)

In the section dealing with multiplication of integers, besides algorithms appropriate to numbers written in the positional decimal system, Mizraḥi presents several methods of oral calculation. The origin of these methods is not clear. Some of them can be found in earlier Hebrew mathematicians, such as Ibn Ezra's *Sefer Hamispar* (the thirds method) and Comtino's *On Reckoning and Measurement* manuscript (the thirds method and the fifths method). These authors, however, present the methods without justification.[29] Unlike his predecessors, Mizraḥi gives a proof for each method he presents.

[27] Preserved in one manuscript in St. Petersburg, Institute of Oriental Studies of the Russian Academy, C 128.

[28] For an analysis, see [Segev, 2010] and [Wertheim, 1896].

[29] The "thirds method" can also be found in Gersonides's *Maʾase Ḥoshev,* but there it appears only for integers that are multiples of 3, and the explanation is based on a numerical example.

The explanations are general and rely on Euclid's geometric propositions from the *Elements*, Book II, interpreted arithmetically. Mizraḥi sometimes quotes Euclid geometrically from Ibn Tibbon's Hebrew translation,[30] but at other times, as here (like Comtino, his teacher), he prefers an arithmetic-algebraic formulation. In this context, it is clear that he interprets Euclid's propositions as dealing with numbers.

Another way to multiply a [two-digit] number by itself: if it has a third, take the third, multiply it by itself, and move the result one [decimal] level higher. Subtract from it the third's square. What remains is the result of multiplying the number by itself [the square of the number].

As an example, if we want to multiply 24 by itself, we take one third of it: 8. Multiply it by itself: 64. Move the result one level higher: 640. Then subtract 64 from it, and what remain are 576. This is the square of 24.

If the number does not have a third, take the closest number which does have a third, either larger or smaller than the given number, and proceed as before. Then, in the case that the number which has a third was less than the given number, add to the result the number which has a third and the given number; in the case that the number which has a third was more than the given number, subtract them from the result.

The example with the number [having a third] which is less than the given number is 25, because the closest number which has a third is 24. We take its third and multiply it by itself as before and we get 576. We add the given number and the number which has a third and we get 625, which is the square of 25. In the example with a number [having a third] which is more than the given number, we choose 23. We take the closest number which has a third, 24, proceed as before and we get 576. Then we subtract from it the given number and the number which has a third, which is 47; the result is 529 which is the square of 23. The method can be used for all numbers.

. . .

[This way of] taking the square of the number's third and moving the result one level higher, then subtracting from it the third's square—the reason for it is known, and is explained by Euclid's *Elements*, Book VIII. For any two square numbers, the ratio of one to the other is equal to the ratio of their sides multiplied by itself.[31] It follows that the ratio of the number's third squared to the number squared is equal to the ratio of the number's third to the number itself, all squared. And the ratio of the number's third to the number is a third, and a third multiplied by itself is a ninth. So, necessarily, the ratio of the number's third squared to the number squared is a ninth. It follows that the square of the given number will be nine times the number's third squared. Raising the third's square one level, we will get ten times the number's third squared. Subtracting from it one third squared, the remainder will be equal to the square of the given number, since it is equal to nine times the number's third squared, because any two magnitudes whose ratio is one are necessarily equal according to Euclid's *Elements*.[32]

[30] Moses ibn Tibbon translated *The Elements* from Arabic into Hebrew in 1270.

[31] *Elements* VIII.5. Euclid's proposition is a more general one, and Mizraḥi infers from it this particular formula.

[32] $10\left(\frac{x}{3}\right)^2 - \left(\frac{x}{3}\right)^2 = 9\left(\frac{x}{3}\right)^2 = x^2$.

. . .

But if the given number doesn't have a third, we use another number that has a third, which is larger or smaller than the given number by one. The reason for this is also clear. It is well known that the difference between the squares of two numbers is equal to the sum of the two numbers multiplied by their difference.[33] And the product of the two numbers is equal to the product of the smaller number by itself added to the product of the smaller number by the difference between the two given numbers, as is written in Book II of Euclid.[34] And the product of the greater number by itself is equal to the product of the greater number by the smaller one, added to the product of their difference by the greater number, for the same reason.[35] It follows necessarily that the greater number multiplied by itself is equal to the sum of three multiplications: the product of the smaller number by itself, the product of the greater number by the difference, and the product of the smaller number by the difference.[36] The last two products are equal to the product of the difference by the sum of the two given numbers, according to the previous argument. So, necessarily, the greater number multiplied by itself is equal to the smaller number multiplied by itself added to the difference multiplied by the sum of the two numbers.[37]

Therefore, if the required number is greater by one than a number which has a third, it is necessary to add the given number and the number which is smaller than it by one and the square of the number which has a third. The sum is the required square. If the required number is greater by two than a number which has a third, add the given number and the number which is smaller than it by two, multiply the result by two and add to it the square of the number which has a third; this is the required square. If the required number is greater by three, multiply by three the sum of the given number and the number which is smaller than it by three and add it to the square of the number which has a third. And so on in all cases.[38]

But if we work with thirds, we have only numbers which are greater or smaller than the required number by one, for if a number is greater by two than a number which has a third, the same is smaller by one than another number which has a third. And that is the reason the ancients chose to use the thirds ... so there will be no need to multiply the sum of the given number with the number which has a third.

But if we use the fifth (or other parts) there will be cases in which it will be necessary to multiply the sum of the two numbers. I therefore wonder why they chose the fifth part, and not any other, when this method is common to all parts.

Summing the first *n* integers and cubes (from Article I, part 1)

In the section on multiplication, Mizraḥi presents the sums of several kinds of arithmetic and geometric progressions, as well as of squares and cubes of the first *n* integers. The sums and

[33] *Elements* II.5, $b^2 - a^2 = (b + a)(b - a)$.

[34] *Elements* II.3, $ba = a^2 + a\,(b - a)$.

[35] *Elements* II.2, $b^2 = ba + (b - a)\,b$.

[36] From the previous two results, $b^2 = ba + (b - a)\,b = \Big(a^2 + a\,(b - a)\Big) + b\,(b - a)$.

[37] From the above, $b^2 = a^2 + (b + a)\,(b - a)$. Note the similarity to *Elements* II.5 above, which is brought up explicitly but is not used in the argument.

[38] $(a + n)^2 = ((a + n) + a)\,n + a^2$. This could also be obtained directly from *Elements* II.6.

their proofs do not use any algebraic terminology or symbolism. Some proofs are based on figurate numbers, but others, like the example below, have a more abstract character, and present original arguments not found in the work of previous authors. Mizrahi's use of the "recursive" identities like $\frac{1+2+\ldots+v}{v+1} = \frac{1+2+\ldots+(v-1)}{n} + \frac{1}{2}$ or $\frac{1^3+2^3+\ldots+n^3}{1+2+\ldots+n} = \frac{1^3+2^3+\ldots+(n-1)^3}{1+2+\ldots+(n-1)} + n$ clearly illustrates some original pre-inductive reasoning.

But the way the ancients used to sum the natural numbers, that is to multiply the last number by its half added to half, is self-evident. For if you take any number and divide the sum of all its preceding numbers by the number itself you will receive the quotient of the sum of all the numbers preceding the preceding number divided by the preceding number when you add to this quotient a half.[39] So the quotient of one divided by two is half, and the quotient of the sum of one and two divided by three is one integer, and the quotient of the sum of one, two, three divided by four is one and a half, and the quotient of the sum of one, two, three, four divided by five is two, and so on. We always add half.

Therefore, when you want to know the sum of all natural numbers from one to any number, because the quotient of the sum of the preceding numbers divided by the last number will be equal to the sum of the halves added in each step, as we have seen before, we must count the steps from one to the given number. Up to two we have half, and up to three we have one, and up to four we have one and a half, and so on until the last number. And so the quotient of the sum of the preceding numbers divided by the last number will be equal to the former result, and if we multiply this former result with the last number, we receive the sum we wanted.

. . .

But the other kind, the cubes of natural numbers, the ancients knew a way to sum them as well: take the cubic root of the last cube, and follow the former way to sum the natural numbers which are the cubic roots of those cubes, and save the result. Then multiply the saved result by itself; this is the sum of the given cubes.[40]

For example, if you want to know the sum of 1, 8, 27, 64, the cubes of 1, 2, 3, 4, take the cubic root of 64, which is 4, multiply it by half the number of steps [i.e., the given cubes] added to half, you receive 10. Multiply 10 by itself, the result is 100, and this is the sum of the numbers 1, 8, 27, 64.

. . .

The way they used to add the cubes of natural numbers by multiplying the sum of their cubic roots by itself—its reason is known as well. For if you take the quotient of the sum of the cubes and the sum of their cubic roots, the result is larger than the quotient of the sum of the cubes previous to the last cube and the sum of their cubic roots by the last cubic root.[41] And so on, each quotient is greater than the previous quotient by the last cubic

[39] $\frac{1+2+\ldots+v}{v+1} = \frac{1+2+\ldots+(v-1)}{n} + \frac{1}{2}$.

[40] $1^3+2^3+\ldots+n^3 = (1+2+\ldots+n)^2 = \left(\frac{n(n+1)}{2}\right)^2$.

[41] $\frac{1^3+2^3+\ldots+n^3}{1+2+\ldots+n} = \frac{1^3+2^3+\ldots+(n-1)^3}{1+2+\ldots+(n-1)} + n$.

root until we reach the first cube [1^3]. It means that the numbers added at every step are the natural numbers.

We skip Mizraḥi's verification of the statement expressed in the previous footnote for sums up to the first four cubes, a verification very similar to what follows.

Therefore the ratio of the cube of 1 and its cubic root [1] is 1 (namely, they are equal), and the ratio of 1, 8 and the sum of their cubic roots [$1+2$] is 3 (namely 3 times, being the former result 1, to which we add 2), and the ratio of 1, 8, 27 and the sum of their cubic roots [$1+2+3$] is 6 (namely, 6 times, being the former result 1, 2, to which we add 3), and the ratio of 1, 8, 27, 64 and the sum of their cubic roots [$1+2+3+4$] is 10 (namely, 10 times, being the former result 1, 2, 3, to which we add 4), and so on, always in the same way.[42]

So if we want to add the cubes of the [natural] numbers, as many as they are, we take the cubic root of the last cube and we multiply it by its half added to half, and the result is the sum of the cubic roots of the given cubes. Next we want to know the number of steps from the first cube to the last one, and we take 1 for the first step [cube], 2 for the second, 3 for the third, 4 for the fourth and so on until we reach the last cube. Next we add all these numbers and this is the quotient of the sum of cubes and the sum of their cubic roots. We therefore multiply the sum of the cubic roots by the sum of the natural numbers (i.e. the sum of the steps) and we receive the sum of the given cubes.

Simple fractions (from Article I, Parts 1 and 2)

The concept of simple fractions underwent many changes over the centuries before it reached the form we use today. The ancient Egyptians used mostly unit fractions, while the Babylonians used the sexagesimal system not only for integers but also for numbers less than one. Fractions that are written with the numerator above the denominator (but without the horizontal line in between) can be found starting from the ninth century in several Arabic texts, and even earlier in Indian texts.[43] The algorithms used for different calculations were often cumbersome and the definitions inconsistent. In contrast, Mizraḥi presents a well-articulated concept of fractions and sometimes argues with previous authors, as in the following excerpt. The algorithms for calculations presented by Mizraḥi (most of them still in use today) are meticulously explained using propositions from Euclid or Nicomachus. Note, however, that unlike the Greeks, Mizraḥi does not seem to distinguish ratios from fractions and quotients.

42

$$
\begin{aligned}
&\frac{1^3+2^3}{1+2} = \frac{1^3}{1} + 2 = 1+2 \\
&\frac{1^3+2^3+3^3}{1+2+3} = \frac{1^3+2^3}{1+2} + 3 = 1+2+3 \\
&\frac{1^3+2^3+3^3+4^3}{1+2+3+4} = \frac{1^3+2^3+3^3}{1+2+3} + 4 = 1+2+3+4 \\
&\vdots
\end{aligned}
$$

43 The origin of writing fractions vertically probably has to do with the way the division operation was written. The horizontal bar used in the writing of fractions can be found for the first time in the work of al-Ḥaṣār from the twelfth century. We can also find this vertical notation in Indian writings from the sixth century [Djebbar, 1992; Mazars, 1992].

In this short excerpt Mizraḥi compares a "relative" notion of fraction (one whole number with respect to another) and an absolute one (parts of the actual unit). He endorses only the former.

I saw some contemporaries who think that the wholes are numbers from one onward, such as one or two or three or any other number, and that fractions are smaller than one, such as half of one or a third or a quarter, or another part of one. Regarding the issue of the parts of one they are right only in one aspect, that is relatively, but they were wrong in thinking of the relative as if it were absolute. That is, they were right when dealing with parts of the whole one, but this "one," you should know that it can be any number we name a whole, because any number can be thought of as a whole in relation to its parts. For example, two with respect to eight is a quarter of eight, and indeed we call the two a quarter of eight when the eight is the whole and the two is the part, the eight being four times two. It [the eight] is the one [whole] whose fractions we calculate, but it [the eight] is not the real one which is indivisible. Now a general definition should include all special cases, whereas if there are two kinds of numbers, wholes and fractions, and if the fractions are parts of the indivisible one, then the definition of the number as the sum of units is contradicted, because the real one [unit] is not a number and neither are its fractions.

In the next excerpt Mizraḥi proves the standard fraction multiplication algorithm.

And I say that the general way to multiply [all kinds of fractions] is to multiply the numerator with the numerator and save the result, then multiply the denominator with the denominator[44] and save the result, and the ratio of the first number you saved to the second one is the fraction or fractions that result from the multiplication.

. . .

The reason for this method, to multiply the numerator with the numerator and the denominator with the denominator, is evident if we know five propositions.

The first of them is that if you take any fraction and multiply its numerator and its denominator by the same arbitrary number, the ratio of the results will be the given fraction relating the original numerator and denominator. The same holds for division. That is, if you divide the numerator and the denominator in arbitrarily many equal parts, and you take an equal number[45] of parts of the numerator and the denominator, and you set the ratio of these parts of the numerator to the parts of the denominator, then this new fraction will be equal to the given fraction relating the undivided numerator and denominator. And the reason is in the *Elements* of Euclid, Book V,[46] stating that equimultiplied parts relate to each other as their parts relate to each other. The proof is common to both.

The second proposition is that saying that we multiply one fraction by another is like saying that we take a fraction from another. This is obvious.

[44] Mizraḥi uses the terms "quantity" and "quality" for the numerator and denominator, respectively.

[45] We read "number" where the text has "parts." According to this reconstruction, Mizraḥi tries to combine the statements about multiplication and division into one.

[46] *Elements* V.15: $\frac{a}{b} = \frac{ac}{bc}$ or $\frac{a}{b} = \frac{a/c}{b/c}$.

The third proposition is that for any number which is multiplied by another, the first number to the product is a fraction derived from the name of the second number.[47] For example, the number two, when multiplied by three, is six. The two will be a third of six, where the name "third" is derived from "three," which multiplies the two. This is explained also in the book of Nicomachus.

The fourth proposition is that if you want to take a part from a fraction, take this part from the numerator of the fraction, and the ratio of this to the denominator of the given fraction is the part taken.[48] For example, if you want to take a third from nine tenths, take a third from nine, which is three, relate them to the tenths, and they make three tenths. This is always so, and is obvious.

The fifth proposition is that if you want to multiply a fraction however many times, take the ratio of the multiplied numerator to the denominator, and the result is the product of the multiplied fraction.[49] For example, if you want to multiply two ninths by three, we multiply two by three, which is six, and relate them to the ninths, yielding six ninths. This is the result of two ninths multiplied by three, and this is also obvious.

Now that you've understood these propositions, the reason for this multiplication is as clear as can be. If we multiply, for example, three quarters by two ninths, it is the same as taking three quarters of two ninths, according to the second proposition. If we could divide the numerator of the two ninths by four to get its quarter, we would take this quarter and relate it to the denominator, and the result would be the quarter of the two ninths according to the fourth proposition. Then we would multiply the numerator by three, and get three quarters of two ninths according to the fifth proposition.

But because the numerator of the two ninths is not divisible by four, we have to multiply it by four (which is the denominator of the three quarters), so that the product of two and four becomes divisible by four. And having done that, we have to multiply also the nine (which is the denominator of the two ninths), by the denominator four, so that the ratio of the products of two and of nine by four equals the two ninths according to the first proposition. We now take the ratio of a quarter of the resulting numerator (which is the product of the numerator by the denominator) and the resulting denominator (which is the product of the denominator by the denominator), and this is a quarter of the two ninths according to the fourth proposition. Then we multiply its numerator by three, and this will be three quarters of two ninths, according to the fifth proposition.

Word problems (from Article III, part 1, chapter 1)

In the third article of his book Mizraḥi presents about 100 problems and their solutions. He classifies them in two categories: number problems and geometric problems. The so-called number problems actually deal with a wide range of topics and present various techniques for solving them. Many of them are standard problems that can be found in earlier books, but Mizraḥi's treatment tends to be deep and well reasoned.

The questions selected here are two mathematical riddles about combining and comparing the money of two or three people. Mizraḥi solves one with proportion theory and the other

[47] If $ab = c$, then $\frac{a}{c} = \frac{1}{b}$.
[48] $\left(\frac{a}{b}\right)/c = \frac{a/c}{b}$.
[49] $\frac{a}{b} \times c = \frac{a \cdot c}{b}$.

with algebraic-like manipulations (other authors solve them with ad hoc manipulations or double false positioning).

Question [15]: A man told his friend, if you give me one, I will have as much as you. His friend answered: if you give me one, I will have twice as much as you.[50] How much does each one have?[51]

The answer is that this question is also misleading, because it posits that which determines for that which is determined.[52] Here we need the ratio of the two [coins] set apart (one from each man) to the money of both, rather than the ratio of the two [coins] set apart together with the money of the first to the money of the second with one subtracted. However, because the latter ratio determines the ratio between the number two and all the money [of both men], he took one for the other. This is self-evident if we consider the money of both men together as one amount, and set apart two, one from the first man and the other from the second, which are what each said he would give to the other. The money of both men is then divided into three parts: the two set apart by both, the money of the first less one, and the money of the second less one.[53] From this we can determine the answer.

The first man saying to his friend: "if you give me one, I will have as much as you" is the same as if he said: "if I add to my money[54] the two [coins] set apart (the one you want to give me and the one I want to give you), the result will be the same as the amount you have less one. Therefore the money of the first man together with the two set apart, which are two of the three parts composing all the money, will necessarily be half of the money composed of the three parts, because the two former parts together equal the third part.[55]

The second man saying to his friend: "if you give me one, I will have twice as much as you," is the same as if he said: "if I add to my money the two set apart, the sum will be equal to twice the rest of your money." Therefore the money of the second added to the two set apart will necessarily equal two thirds of the money composed of the three parts, as the two former parts are twice the third part.[56]

So the money of the first man is a third of all the money composed of the three parts, and we have already seen that half of all the money is equal to the money of the first man together with the two set apart. Therefore the difference between the half and the third, which is a sixth [of all the money], equals two, which is what we wanted to find: the ratio between the two set apart and all the money. [57]

So we apply ratios:[58] if one sixth equals two, how much is the whole? This yields twelve. We have already said that the money is composed of three parts: the two set

[50]We omit a variation where the parameters are 2 and 100 instead of 1 and 2.

[51]For a version of this problem by Fibonacci, see Chapter 1, section II-3-2.

[52]This is oddly phrased. In what follows Mizraḥi suggests that to solve the question, one must transform the data so that it fits a standard Rule of Three procedure. If x is the money of the first man and y the money of the second, we are given the ratios $(x+1):(y-1)$ and $(y+1):(x-1)$, whereas we need the ratio $2:(x+y)$.

[53]By considering the parts $a=2$, $b=x-1$ and $c=y-1$, we have as data the ratios $(a+b):c$ and $(a+c):b$, and require the ratio $a:(b+c)$.

[54]From here on the money of the partners is considered as $x-1$ and $y-1$, instead of x and y.

[55]$(x-1)+2=(y-1)$, so $(x-1)+2=\frac{1}{2}\left((x-1)+2+(y-1)\right)$.

[56]$(y-1)+2=2\,(x-1)$, so $(y-1)+2=\frac{2}{3}\left((y-1)+2+(x-1)\right)$.

[57]$(x-1)=\frac{1}{3}\left((x-1)+(y-1)+2\right)$ and $(x-1)+2=\frac{1}{2}\left((x-1)+(y-1)+2\right)$, so $2=\frac{1}{6}\,(x+y)$.

[58]This is Mizraḥi's term for what is known as the Rule of Three.

apart, the money of the first man, and the money of the second, and that the sum of the two and the money of the first is half of all the money. So, necessarily, the money of the first man together with the two set apart is six, which are half of twelve. And if we take away the two added to the money of the first, there remain four, the money of the first. So the money of the second will be six. And if we give to each the coin we took, the first will have five and the second will have seven, and this is the money they each have.

Question [35]: Reuven, Simon and Levi went to the fish market and found a fish. Reuven said to his friends: If I gave all my money, and each of you gave half of yours, we could buy the fish. Simon answered and said: If I gave all my money, and each of you gave a third of yours, we could buy the fish. Levi answered and said: If I gave all my money, and each of you gave a quarter of yours, we could buy the fish. What is the ratio between their money, that is, the ratio of each to each?[59]

The answer is that this question is misleading, because the problem omits the ratio between the money of each and the others', stating instead facts that determine them. So if we obtain what is determined by the facts stated in the problem, we will arrive at the answer.

We say that the statements of Reuven and Simon determine that half of Simon's money equals two thirds of Reuven's and one sixth of Levi's. Therefore Simon's entire money equals one and a third of Reuven's and a third of Levi's. That is because all of Reuven's money together with half of the others', as Reuven says, equals all of Simon's money together with a third of the others', as Simon says. Therefore, what Simon added to Reuven's statement—a half of his [Simon's] money—equals what he removed from Reuven's total—two thirds of Reuven's—and to what he removed from half of Levi's money—one sixth of Levi's.[60]

In the same way, according to Reuven's and Levi's statements, it is determined that half of Levi's money equals three quarters of Reuven's and one quarter of Simon's. Therefore, in the same ratio, a third of Levi's money equals half of Reuven's and a sixth of Simon's.[61]

We already know that Simon's money equals one and a third of Reuven's and a third of Levi's. Therefore Simon's money equals one and five sixths of Reuven's and one sixth of Simon's own. We remove one sixth of Simon's which is common [to both sides], and remain with five sixths of Simon's money being equal to one and five sixths of Reuven's. According to the same ratio, all of Simon's money equals twice Reuven's and its fifth.[62]

Therefore it is determined that if Reuven had five, Simon would have eleven. We already saw that half of Levi's money equals three quarters of Reuven's and one quarter

[59]Here, if a is the money of Reuven, b is Simon's money, c is Levi's money, and d is the price of the fish, we can write three equations for this problem: $a+\frac{1}{2}b+\frac{1}{2}c=d$, $b+\frac{1}{3}a+\frac{1}{3}c=d$, and $c+\frac{1}{4}a+\frac{1}{4}b=d$. For a version of this problem by Fibonacci, see section II-2-2 in Chapter 1. For Levi ben Gershon's abstract version, see section I-6 in this chapter.

[60]$a+\frac{1}{2}b+\frac{1}{2}c=b+\frac{1}{3}a+\frac{1}{3}c$, so the excess of the right hand over the left hand, which is $\frac{1}{2}b$, equals what the right hand removes from the left hand, namely, $\frac{2}{3}a+\frac{1}{6}c$. We rescale by two and obtain $b=1\frac{1}{3}a+\frac{1}{3}c$.

[61]By comparing the second and third identities, we get $\frac{1}{2}c=\frac{3}{4}a+\frac{1}{4}b$. Rescaling the last identity by two-thirds, we obtain $\frac{1}{3}c=\frac{1}{2}a+\frac{1}{6}b$.

[62]By substitution, $b=1\frac{1}{3}a+\frac{1}{3}c=1\frac{1}{3}a+\left(\frac{1}{2}a+\frac{1}{6}b\right)$. Rearranging yields $\frac{5}{6}b=1\frac{5}{6}a$, or $b=2\frac{1}{5}a$.

Fig. I-6-1. The Tel Aviv *Ralbag* street honors Levi ben Gershon. Under his dates, the sign reads, “Acronym for the name of Rabbi Levi ben Gershon: Philosopher, Mathematician, Astronomer, and Commentator on the Bible.” Photograph by Phyllis Katz.

of Simon’s. This ratio determines that all of Levi’s money will be one and a half of Reuven’s and half of Simon’s. So it is determined that if Reuven had five and Simon had eleven, as we have already seen, Levi would have thirteen. This determines the price of the fish to be seventeen.

6. LEVI BEN GERSHON, *MAʾASE ḤOSHEV* (*THE ART OF THE CALCULATOR)*

Levi ben Gershon (1288–1344), who lived his entire life near Orange in Provence, was one of the most prominent medieval Jewish scientists, besides also being a rabbi, philosopher, and biblical commentator. In Jewish circles, he is known by the acronym *Ralbag*, and in Latin circles as Gersonides [Fig. I-6-1]. He wrote numerous commentaries on the Hebrew Bible, a book on logic, four mathematical treatises, and a major philosophical work, *Milḥamot Adonay* (*Wars of the Lord*), which includes a section on trigonometry as part of a longer section on astronomy, in which he criticizes some of the ideas of Ptolemy. During Levi’s lifetime, the Jews in Provence (about 15,000 out of a total population of 2,000,000) were under the protection of the pope, then residing in Avignon. Levi was well regarded by the Christian community as a scientist, and some of his works were translated into Latin during his lifetime, although Levi himself probably did not know the language.[63]

Maʾase Ḥoshev[64] is Levi’s first book on mathematics and is dated 1321, with a second edition the following year that had minor changes in organization and presentation.[65] The work exists today in twelve manuscripts, nine of the first edition and three of the second.

[63] See [Freudenthal, 1992] for more details on Levi’s life and work.

[64] The title means, literally, “Thoughtful Application,” a biblical reference (Exodus 26:1 and 28:6) to the kind of work used in building the Tabernacle. In the context of that building, this is work that requires thought, planning, and calculation rather than only technical craftsmanship.

[65] The discovery of the second edition is discussed in [Simonson, 2000b].

The oldest manuscript is *Parma 2271*, a first edition manuscript, estimated to have been written in Provence late in the fourteenth century. The critical edition and German translation by Gerson Lange of 1909 was based primarily on the Vienna manuscript of 1462, also of the first edition. *Maʾase Ḥoshev* is in two parts followed by a large collection of problems. The first part is theoretical, containing 68 theorems and problems in Euclidean style dealing with arithmetic and combinatorics, with some of the proofs being accomplished by a form of mathematical induction [Rabinovitch, 1970]. Its style is reminiscent of the arithmetic of Jordanus de Nemore (see section II-2-5 of Chapter 1). The second part contains algorithms for calculation and is subdivided into six sections.

Introduction

Signed Levi ben Gershon. In order to acquire a complete ability to do practical crafts, one needs to know the craft—a craft with knowledge of technique, and why to use a particular technique. The practical part of the craft of numbers is one of the practical crafts. So it is clear that it is worthwhile to investigate its theory. Another reason why it is mandatory to investigate this craft and its given theory, is that it is clear that this craft encompasses many different kinds, and each and every kind encompasses many diverse topics, so that you might think they are not all part of the same kind. Because of this, it is clear that you will not complete your acquisition without knowledge of the theory, except with great difficulty. However, with knowledge of the theory, it is possible to complete your acquisition with ease. This is so, because he who knows the theory can, with a single view, understand the practical characteristics of each of the many kinds that the craft encompasses, and he who masters the theory will require just a single view in place of many views for the various topics. Accordingly, we see fit to present the manipulation of numbers and its theory, for our benefit. Along this line of thought, I have divided this book into two sections.

Section One focuses on the principles needed to understand this craft. Section Two focuses on the practical craft of manipulating numbers, one kind at a time, and the explanations. And since this book focuses on application and investigation, it is called *Maʾase Ḥoshev*.

However, as far as the instruction in this book is concerned, it is appropriate that he who concerns himself with it should already understand the 7th, 8th, and 9th books of Euclid. It is not our intention to repeat his words in our book, but instead to assume his principles in our development, as they were proved there.

. . .

Arithmetic with proofs (from part I)

This section contains many elementary and less elementary arithmetical facts and problems. Levi endows each fact or problem with a proof in his contemporary adaptation of the Euclidean style, using letters to refer to the quantities discussed but not operating on these letters algebraically. Some proofs depend on recursive calculation and inductive procedures.

1. The product resulting from the multiplication of two numbers one with the other counts every part of the first number as many times as there are ones in the second number.[66]

2. When you have two given numbers, and one is partitioned into some number of parts, then the product of the first number with the second equals the products of each part of the first number with the second number, all added together.[67]

The given numbers are *AB* and *C*, with *AB* divided into the parts *AE*, *ED*, *DB*. I assert that the product of *AB* with *C* is equal to the sum of the products of *AE* with *C*, *ED* with *C*, and *DB* with *C*. In the product of *AE* with *C*, the factor *C* occurs as often as the units in *AE*; also the factor *C* occurs in the product of *DE* with *C* as often as the units in *DE* and in the product of *DB* and *C* as often as the units in *DB*. In *AE*, *ED* and *DB* together, however, are just as many units as in *AB*. Thus in the sum of the products, *C* is a factor as often as the units in *AB*. Therefore, in the product of *AB* with *C*, *C* occurs as often as the units in *AB*. Thus the product of *AB* with *C* is equal to the sum of these products.

. . .

9. When you multiply one number by a number built[68] from two given numbers and the result is something, then if you multiply a number built from any two of these three numbers by the third number, the result will be the same.

10. When you multiply one number by a number built from three given numbers and the result is some number, then if you multiply any one of these numbers by the number built from the remaining three, the result will be the same.[69]

Multiply *A* by the number built from *CDE*, giving *FG*. I am saying that if you multiply *D* by the number built from *ACE*, the result will also be *FG*. The proof is that we divide *FG* into parts corresponding to the numbers *CDE* and these parts are *FI*, *IL*, *LG*, then the number of parts is like the number of units in *A*. And, each one of the parts *FI*, *IL*, *LG* counts *D* like the amount of the product *CE*. And this is explained by what came earlier [theorem 9]. Now *FG* counts *D* as much as all its parts together, but all its parts together count *D* like the value of the product *CE* multiplied by *A*. Therefore, *FG* in total counts *D* like the number built from *ACE*. And therefore, the product of *D* with the number built from *ACE* is *FG* also.

And similarly it can be explained that whichever of these numbers is multiplied by the number built from the rest, the result will be *FG*. And like this, the progression can be understood without limit. What I mean to say is that if you multiply a certain number by a number built from four numbers, and let that be some number, then if you multiply any one of these numbers by the number built from the remaining numbers, then the result will be the same. And therefore, the number resulting from the multiplication of any number by

[66] That is, $ab = a + \ldots + a$, b times.

[67] $(a_1 + a_2 + \ldots + a_n)\, b = a_1 b + a_2 b + \ldots + a_n b$.

[68] The word translated as "built" is *murkav*—defined by Levi as the product or composite of the two numbers. This theorem asserts the associativity and commutativity of multiplication of three numbers. The proof uses theorem 1.

[69] This theorem generalizes theorem 9 to four numbers, and then uses a form of mathematical induction to generalize to n numbers.

the number built from the rest counts the number like the value of the number built from the rest.

. . .

17. If you subtract from a given number a given part or given parts and take from the remainder another given part or given parts and so forth, then the final remainder will be the same and the sum of the pieces taken will be the same, no matter in what order the parts are taken.

Let *A* be the given number, the parts having denominators *B*, *C*, *D*, and let the *B*th part of *A* be subtracted, then *E C*th parts from the remainder, and *F D*th parts from the next remainder. I say that the sum of the *B*th part of *A*, *E C*th parts of the remainder, and *F D*th parts of this remainder is equal to *F D*th parts of *A* added to the *B*th part of the remainder and *E C*th parts of that remainder.[70]

For the proof we set *G* to be the number preceding *B*, and further let the units of the sum of the numbers *E* and *H* equal *C* and the numbers *F* and *I* equal *D*. Let the *B*th part of *A* equal *P*, and the remainder of that from *A* be *J*; let *E C*th parts of *J* be *K* and let *L* be the next remainder; finally let *F D*th parts of *L* equal *M*, with *N* being the final remainder. Also, let *F D*th parts of *A* equal *Q*, with remainder *R*, the *B*th part of *R* equal *S*, with remainder *T*, and *H C*th parts of *T* equal *U*, with remainder *V*. We claim that *N* equals *V*. The proof: Because *P* is equal to the *B*th part of *A*, therefore *A* has as many parts equal to *P* as *B* has units. But *G* is equal to the number preceding *B*, so *J* has as many parts equal to *P* as *G* has units. Also, we know that *A* is to *J* as *B* is to *G*, since the product of *B* by *P* is *A* and the product of *G* by *P* is *J*. Now set the *C*th part of *J* equal to *W*, so *K* has as many parts equal to *W* as *E* has units, and therefore *L* has as many parts equal to *W* as *H* has units. It is shown similarly that *J* is to *L* as *C* is to *H* and therefore that *L* is to *N* as *D* is to *I*. So it is proved that the ratio of *A* to *N* is composed of the ratios of the numbers *B*, *C*, *D* to the numbers *G*, *H*, *I*.[71] Similarly, it is proved that the ratio of *A* to *V* is composed of the ratios of the numbers *D*, *B*, *C* to the numbers *I*, *G*, *H*;[72] but this ratio is equal to the ratio composed of the numbers *B*, *C*, *D* to the numbers *G*, *H*, *I*, so the ratios of *A* to *N* and to *V* are the same; so *N* equals *V*. Therefore, it must also be true that the sum of the numbers *P*, *K*, *M* equals the sum of the numbers *Q*, *S*, *U*. The difference between *A* and *N* is equal to the sum of *P*, *K*, *M*, and the difference between *A* and *V* is equal to the sum of *Q*, *S*, *U*; but it has been shown that *N* equals *V*, so must the sums of *P*, *K*, *M* and *Q*, *S*, *U* be equal.

26. When you add, beginning with one, consecutive numbers, and if the number of these is even, then the sum is equal to the product of half the number with the number following the last one.[73]

[70] $\frac{1}{B}A+\frac{E}{C}\left(A-\frac{1}{B}A\right)+\frac{F}{D}\left[A-\frac{1}{B}A-\frac{E}{C}\left(A-\frac{1}{B}A\right)\right]=\frac{F}{D}A+\frac{1}{B}\left(A-\frac{F}{D}A\right)+\frac{E}{C}\left[A-\frac{F}{D}A-\frac{1}{B}\left(A-\frac{F}{D}A\right)\right].$

[71] Since $B:G=A:J$, $C:H=J:L$, and $D:I=L:N$, the composition $(B:G)(C:H)(D:I)=(A:J)(J:L)(L:N)=A:N$.

[72] Since $D:I=A:Q$, $B:G=Q:S$, and $C:H=S:V$, the composition $(D:I)(B:G)(C:H)=(A:Q)(Q:S)(S:V)=A:V$.

[73] $1+2+\ldots+n=\frac{n}{2}(n+1)$, where n is even.

Let the numbers be A, B, C, D, E, F,[74] with the number following F being G, and A being one. I say that the sum of A, B, C, D, E, F is equal to the product of half their number with G. For proof, since A is one, the sum of F and A is G. Since the difference between B and one is equal to the difference between F and E, namely one, the sum of B and E is G. Also the difference between C and one is equal to the difference between F and D, so the sum of C and D is also G. Thus in the sum of A, B, C, D, E, F, the factor G occurs as often as half of the number, since the sum of each pair of these numbers is equal to G.

27. When you add consecutive numbers beginning with one, and if the number of these is odd, then the sum is equal to the product of the middle number with the last number.

28. When you begin a sequence of consecutive numbers with one and add together an odd number of terms, multiply half of the last number with the number following it. This product is equal to the sum of the numbers.[75]

30. When you add the sum of the consecutive numbers from one up to a given number to the sum of the numbers up to the number following that number, then the total is equal to the square of the number following the given number.[76]

When you add the sum of the numbers A, B, C, D, E to the sum of the numbers A, B, C, D, E, F, with A being one, then the total is equal to the square of F. To prove this, let G be the number following F. It has already been proved that the sum of A, B, C, D, E is equal to the product of half of E with F [theorem 28] and that the sum of A, B, C, D, E, F is equal to the product of half of F with G [theorem 26]. But the product of half of F with G is equal to the product of half of G with F, since the factors are in the same proportion. So if you add the sum of A, B, C, D, E to the sum of A, B, C, D, E, F, this is the sum of the product of half of E with F and the product of half of G with F, which is equal to the product of half the sum of E and G with F. But since the sum of E and G is equal to twice F, its half is equal to F. So the sum of the sum of A, B, C, D, E with the sum of A, B, C, D, E, F is equal to the product of F with F, that is, the square of F.

38. When you multiply a given number minus one third of the number that precedes it by the sum of the consecutive numbers from one through the given number, the result equals the sum of the squares of the consecutive numbers from one through the given number.[77]

The following two theorems present a proof in inductive style of the formula for the sum of the cubes of consecutive integers beginning with 1. Notice that the inductive step is proved first.

[74] Levi proves many of the following results by the method of generalizable example, since he has no way of representing an arbitrary integer. Given that the letters of the Hebrew alphabet can represent numbers, it is not entirely clear whether one should translate the beginning of the proofs by using letters or by using the numbers they might represent. We decided to use letters when Levi specifically notes that A (or aleph) is one, as in theorems 26 and 30, while using numbers when Levi does not so note (as in theorems 41 and 42).

[75] $1 + 2 + \ldots + n = \frac{n}{2}(n + 1)$, where n is odd. In theorem 27, Levi shows that the sum of each pair at equal distance from the middle term is equal to twice that term. Then in theorem 28, he notes that twice the middle term is equal to $n + 1$, so the product of the middle term with n is equal to the product of twice the middle term with half of n, that is, "half of the last number with the number following it."

[76] $(1 + 2 + \ldots + n) + (1 + 2 + \ldots + n + (n + 1)) = (n + 1)^2$.

[77] $\left(n - \frac{n-1}{3}\right)(1 + 2 + 3 + \ldots + n) = 1^2 + 2^2 + 3^2 + \ldots + n^2$. We omit the proof, which depends on several earlier theorems.

41. The square of the sum of consecutive numbers from one up to a given number equals the cube of the given number added to the square of the sum of the consecutive numbers from one up to the number before the given number.

Let the numbers be 1, 2, 3, 4, 5. I say that the square of the sum of 1, 2, 3, 4, 5 equals the cube of 5 added to the square of the sum of 1, 2, 3, 4. The cube of 5 is computed by counting 5 once for each unit in the square of 5. But the square of 5 equals the sum of 1, 2, 3, 4 added to the sum of 1, 2, 3, 4, 5 [theorem 30]. So 5 times the sum of 1, 2, 3, 4 added to the sum of 1, 2, 3, 4, 5 equals the cube of 5. But 5 times the sum of 1, 2, 3, 4 added to the sum 1, 2, 3, 4, 5 equals the sum of 5 times 5, that is, the square of 5, and the product of 5 with the sum of 1, 2, 3, 4 and 1, 2, 3, 4, that is, twice 5 times the sum of 1, 2, 3, 4. So the cube of 5 equals the square of 5 plus twice 5 times the sum of 1, 2, 3, 4. Also, the square of the sum of 1, 2, 3, 4, 5 equals the square of 5 plus twice 5 times the sum of 1, 2, 3, 4 plus the square of the sum of 1, 2, 3, 4. Therefore the cube of 5 plus the square of the sum of 1, 2, 3, 4 equals the square of the sum of 1, 2, 3, 4, 5, and this is what we wanted. Finally, we know that one has no number before it. However, its cube equals the square of the sum of the numbers up to it because it is exactly the sum of the number up to it, and hence it is exactly the square of this sum. And this is identical to its cube. This is perfectly trivial.

42. The square of the sum of consecutive numbers from one up to a given number equals the sum of the cubes of the consecutive numbers from one up to the given number.

Let the sum be the sum of 1, 2, 3, 4, 5. I say that the square of the sum of 1, 2, 3, 4, 5 is equal to the sum of the cubes of the numbers 1, 2, 3, 4, and 5. The proof is that the square of the sum of 1, 2, 3, 4, 5 equals the sum of the cube of 5 and the square of the sum of 1, 2, 3, 4 [theorem 41]. But the square of the sum of 1, 2, 3, 4 equals the sum of the cube of 4 and the square of the sum of 1, 2, 3. And the square of the sum of 1, 2, 3 equals the sum of the cube of 3 and the square of the sum of 1, 2. And the square of the sum of 1, 2 equals the sum of the cube of 2 and the square of 1, and the square of 1 equals the cube of 1. Therefore, the square of the sum of 1, 2, 3, 4, 5 equals the sum of the cubes of the numbers 1, 2, 3, 4, and 5. And that is what we wanted to prove.

. . .

53. Problem: To find three numbers such that the first, increased by a given part of the sum of the other two, equals the second, increased by another, also given, smaller part of the sum of the other two, and equals the third, increased by a given third part of the sum of both others that is smaller than the other two parts.[78]

[78]The problem is to find three numbers G, H, I such that $G + \frac{1}{A}(H + I) = H + \frac{1}{B}(G + I) = I + \frac{1}{C}(G + H)$, where $A < B < C$. Levi gives a general, abstract solution to this problem, which also appears in Diophantus's *Arithmetica*, Book I, #24 (see Appendix 2). However, Levi's solution is very different from that of Diophantus. In a "real-world" form, the problem becomes a recreational problem about three men buying a horse; see Fibonacci in section II-2-2 of Chapter 1 for an example, as well as the work of Mizraḥi above. The problem also occurs in the work of al-Karajī around the year 1000 [Woepcke, 1853, p. 95]. Levi solves this under two separate conditions, first with

. . .

58. Problem: To find three numbers such that the sum of the first and third contains the second as a factor as many times as a given number and such that the sum of the second and third contains the first as a factor as many times as a second given number.[79]

Let the given numbers be A, B. We denote the number following A by C; this is the first number. We denote the number following B by D; this is the second number. We take the product of A with B, subtract one and designate this by E; this is the third number. I say that C, D, E are the three sought numbers. We show that the sum of C and E is A times the factor D and the sum of D and E is B times the factor C. Since E is equal to one less than the product of A and B, and C is equal to the sum of one and A, then the sum of C and E is equal to the sum of A with the product of A, B. But the sum of A with the product of A, B is equal to the product of A and D. So the sum of G and E is equal to the product of A and D and therefore is A times the factor D. Furthermore, E is equal to one less than the product of A and B, and D is equal to the sum of B and one, so the sum of D and E is equal to the sum of B with the product of A, B. But the sum of B with the product of A, B is equal to the product of C and B, so the sum of D and E is equal to the product of C and B. The product of C and B contains B times the factor C. So the sum of D and E contains B times the factor C, and the sum of C and E contains A times the factor D. And this is what we wanted to show.

Decimal-sexagesimal subtraction (from part II, chapter 1)

We present an example of a subtraction problem with sexagesimal fractions. In the example, "firsts" mean sixtieths, "seconds" mean 3600ths, and so on. But note that Levi writes the whole numbers out in words while using the Hebrew alphabet numerals to write out the fractions. The discussion ends with the possibility of subtracting larger numbers from smaller numbers in circular contexts (e.g., angles, days of the week).

We want to subtract two hundred six wholes, fifty firsts, 37 thirds from 31 thousand and eighty wholes, 46 seconds, 35 thirds, 47 fourths, 53 sixths.

	0	10	7	9		45				
3	1	0	8	0	0	46	35	47	0	53
		2	0	6	50	0	37			
3	0	8	7	3	10	45	58	47	0	53

$A = 2$ and second with $A > 2$. However, the solutions are the same in each case: $G = (A-2)\,BC + C + B - A$, $H = G + 2\,(B-A)\,(C-1)$, $I = H + 2\,(C-B)\,(A-1)$. We omit Levi's detailed proof that these are the correct values.

[79] The problem is to find three numbers C, D, E so that $C + E = AD$ and $D + E = BC$, with A, B given numbers. As before, Levi gives a general, abstract solution to the problem. However, this problem appeared earlier as a recreational problem about two men finding a purse; see Fibonacci in section II-2-2 of Chapter 1 for an example. The problem also occurs in the ninth-century work of the Indian mathematician Mahāvīra [Rangācārya, 1912, verse 244]. (See Appendix 3.) Note that the problem as stated is an indeterminate one, but Levi only gives one solution. However, when Levi uses this proposition in problem 18 at the end of the book (see below), he shows how to determine a particular solution when one of the unknowns must have a given value.

In the lowest category are the sixths. In that, in the higher row, are 53 sixths. We subtract from that what is in the lower row, but there is nothing, so write 53 in the row of results in the category of sixths. Then we subtract from the 0, that comes next to the 53, what lies below that in the lower row; but there is nothing in the lower row, so write 0 in the row of results in the category of fifths. Then we subtract from the 47 what lies below that in the lower row, but there is nothing, so write 47 in the row of results in the category of fourths. Then we subtract from the 35 in the upper row what lies below in the lower row, which is 37. That cannot be subtracted from 35, so we take one from the category that comes next to 35. So this becomes 60 in this category, which is added to 35, so becomes 95. We subtract 37 from this, so there remain 58. We write that in the row of results in the category of thirds. Now there is left 45 in the following category. We subtract from this what lies below it in the lower row, but there is nothing, so write 45 in the row of results in the category of seconds. There remains now from the 0 to subtract what lies below in the lower row. But 50 is there and we cannot subtract 50 from 0, and also in the category of the whole numbers, that lies next, is nothing that we can bring over. But in the third category from the one in question is a number, namely 8. We take one away from that and put it in the category to the right. We write 7 over the 8 and the 1, that we took away, becomes 10 in the first category. Take one of these into the category of firsts, so there remain 9 in the first category, which we write over the 0. The 1, which was taken, becomes 60 in the category of firsts. From that we subtract 50, so there remain 10, which we write in the row of results in the category of firsts. Now we subtract from the 9 what lies below it in the lower row. That is 6, so there remain 3 which we write in the final row under the units. Furthermore, we subtract from the 7 what lies below in the lower row; there remain 7 because there was 0 there. So we write that in the row of results for the tens. Further we subtract from 0 what lies below in the lower row. That is 2, but we cannot subtract 2 from 0. In the next category is 1, so we take that, which is in this category equal to 10. We write above the 1 a 0, and subtract 2 from the 10. There remain 8, which we write in the row of results in the category of hundreds. Then we subtract from the 0 what lies below in the lower row. But that is 0, so we write 0 in the row of results in the fourth category. Then we subtract from the 3 what lies below in the lower row. That is, however, 0, so we write 3 in the row of results in the fifth category. So the result is thirty thousand eight hundred and seventy three wholes, 10 firsts, 45 seconds, 58 thirds, 47 fourths, 53 sixths. Do the same in similar cases.

Sometimes it can occur in geometric calculations that you must subtract a greater number from a smaller, and indeed that happens in astronomy. You then add the value of the circle circumference, which is equal to 360, to the smaller number, from which you want to subtract, and then you can subtract what you wish, because there is no number in astronomical calculations that is larger than 360. For if it happens that a number will be larger than 360, one takes away that much and uses only the remainder. Such a calculation also appears in the ordinary new moon calculation. If one must subtract a greater number from a smaller number, one adds 7 days to the smaller number, and then you can subtract what you wish, since if the calculator of the new moon obtains a number larger than 7 days, he subtracts off the 7 days and uses only the remainder. Do the same in similar cases.

Mental multiplication technique (from part II, chapter 2)

The basic idea here is a trick for multiplying two digit numbers. It is based on the identity $(a+b)(c-b)+((a+b)-c)\,b=ac$, where b may be an added or a subtracted number. Choosing b such that $a+b$ is a multiple of 10 simplifies the mental calculation.

To make it easier for you, I will give you a number of ways with which to calculate the multiplication of one number by another easily. You already know that multiplying a number of the first rank by a number of the first rank is an easy task, and so is multiplying a "broken" number. By that I mean multiplying a number of the first and second ranks by a number of the first rank.[80] But if you want to multiply one broken number by another broken number, complete one of the numbers to the side that is closest. If you added to this number in order to complete it to the nearest ten, subtract from the other number the amount that you added to the first number, and multiply what is left by the number in your hand, and save the result. If you subtracted from this number in order to complete it, add to the other number the amount you subtracted from the first number, and multiply what remains in your hand by the completed number that is in your hand. After this, look at how much the larger number after the addition or subtraction exceeds the smaller number before the correction. Multiply this excess by the amount you added to one of the numbers, and save the result. This is the second saved value. After this, look at the number that you subtracted. If you subtracted from the big number, then subtract the second saved value from the first saved value, and what remains in your hand is the desired result. If you added to the big number, then add the second saved number to the saved first number, and that is the desired result.

I will give you some examples. We want to multiply 34 by 57. Complete the number 57 to the nearest ten to get 60. Since 60 exceeds 57 by three, subtract three from 34 to get 31. Multiply 31 by 60 to get one thousand eight hundred and sixty, and that is the first saved value. Since 60 exceeds 34 by 26, we multiply 26 by three and that is 78, the second saved value. Since we added to the large number, we add the second saved value to the first saved value and that is one thousand nine hundred and 38, which is the desired result.

In this example of ours, if we lowered 57 to the ten below it, we get 50, and we add 7 to 34 to get 41. We multiply 41 by 50 to get two thousand and fifty, and that is the first saved value. Since 50 exceeds 34 by 16, we multiply 16 by 7 to get 112, and that is the second saved value. Since we subtracted from the big number, we subtract the second saved value from the first saved value, leaving one thousand 9 hundred and 38, which is the desired result.

. . .

Sometimes it will happen that when using this method you will multiply a number by itself, and then this method will make things very easy. For example, you need to multiply 43 by 57. If you complete 43 to 50, you subtract the amount of the completion from 57 to

[80] By a number of the "first rank," Levi means a number less than 10; a number of the "second rank" is a multiple of 10, while a "broken" number is a two-digit number with neither digit equal to 0.

get 50. You will need to multiply 50 by 50 and subtract 7 squared from the result, leaving the desired result. This is very clear from the earlier material at the start of part one of this book. May you understand and discover.

Summing arithmetic and geometric series (from part II, chapter 3)

If you want to add the consecutive numbers from 1 through a given number, take half the square of the given number and add it to half the given number, and that is the desired result. For example, if you want to add one, two, three, four, and so on until ten, including ten, take half the square of ten and half of it to get 55, and that is the desired result. Another way is to multiply this number by half the number that follows it, or half the number by the number that follows it, and that is the desired result. In our example, multiply 10 by half of 11, or half of 10 by 11, to get 55, which is the desired result.[81]

If the numbers follow one another but not [starting with one], that is, if the first is a given number, and the second is twice the given number, and the third is three times that number, and so on until some number [that is, if the numbers are in an arithmetic sequence beginning with the common difference], add up all the numbers until this number in the previous manner, and multiply the result by the first given number, and that is the desired result.[82]

For example, suppose that the first is 7 and the second 14 and the third 21 and the fourth 28, and continue in this way through 9 numbers. You already know that the sum of consecutive numbers from one through nine is 45. Multiply this by 7, which is the first number, to get 315, and that is the desired result.

This is so because the ratio of one to the first is equal to the ratio of two to the second and to the ratio of three to the third and to the ratio of four to the fourth and to the ratio of five to the fifth and to the ratio of six to the sixth and to the ratio of seven to the seventh and to the ratio of eight to the eighth and to the ratio of nine to the ninth. But the ratio of one to its neighbor is equal to the ratio of all to all. Therefore, the ratio of one to seven is like the ratio of all to all. But seven counts one as many times as there are ones in seven, therefore all of these numbers count 45 as many times as there are ones in seven. Therefore, simply multiply the number 45 by 7 and the result is equal to the sum of these numbers. Ponder this.

. . .

If you want to add the squares of consecutive numbers from one through a given number, take the given number less one third of the number that precedes it, and multiply this by the sum of the consecutive numbers through the given number.[83] For example, if you want to know the sum of the squares of the consecutive numbers through 5, since the number preceding five is four, we subtract one third of four, which is 4 thirds, leaving

[81] $1+2+3+\ldots+n=\frac{n^2}{2}+\frac{n}{2}=n\times\frac{n+1}{2}=\frac{n}{2}\times(n+1)$.

[82] $a+2a+3a+\ldots+na=(1+2+3+\ldots+n)\,a$.

[83] $1^2+2^2+\ldots+n^2=\left(n-\frac{n-1}{3}\right)(1+2+\ldots+n)$.

four less a third. Multiply this by 15, which is the sum of the consecutives through 5, and you get 55, which is the desired result.

. . .

If you want to add the cubes of consecutive numbers from one through a given number, take the square of the sum of the consecutive numbers from one through the given number, and the result is what is desired.[84] For example, if you want to know the sum of the cubes of the numbers from one through six, the sum of the consecutives from one through six is 21, and taking its square gives 441, the desired result.

If the numbers [form an arithmetic sequence beginning with the common difference] and there are a given number for which we want to know the sum of their cubes, find the sum of the cubes of the consecutive numbers from 1 up to the given number and multiply this by the cube of the first number; that is the desired result.[85] The reason for this has already been given earlier. For example, suppose the first number is 4, the second 8, and we follow this law for five numbers. We know that the sum of the third powers of the consecutive numbers up to 5 is equal to 225. We multiply this by 64, the third power of the first number. The result is 14400, which is the desired result.

. . .

Levi next gives rules and examples for summing arithmetic sequences whose first term is not equal to the common difference. He then supplies similar rules for their squares and cubes.

If you want to add a given number of cubes of numbers in an arithmetic sequence whose first term differs from the common difference by a second given number, then if the first number is less than the common difference, take the sum of the squares of these numbers and multiply the result by the triple of the second given number, and save the result. Also, multiply three times the square of the second given number by the sum of these numbers, and save this result. Then, multiply the cube of the second given number by the first number, and add the result to the two earlier saved values, and you have in your hand the adjusted first saved value. After this, take the sum of the cubes of the numbers from 1 through the first given number, and multiply it by the cube of the common difference. Finally, subtract the adjusted saved value from this value, and what remains is the desired result.[86]

For example, if we want to add the cubes of seven numbers, where each number is three greater than its predecessor, and the first is two less than three, then we know that the sum of the squares of these numbers is 952. Multiply this by three times two, which is six, to get 5712, and save this. And also, the sum of these numbers is 70. Multiply this by three times two squared, which is 12, to get 840, and save that. Also, multiply the cube of two, which is eight, by 7, to get 56. Add this to the two saved values to get 6608, which is

[84] $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$.

[85] $a^3 + (2a)^3 + \ldots + (na)^3 = a^3 \left(1^3 + 2^3 + \ldots + n^3\right)$.

[86] If d is the common difference, a the first term, and $t = d - a$, then this algorithm depends on the formula $(dk - t)^3 = [(dk - t) + t]^3 - 3\,(dk - t)^2\, t - 3\,(dk - t)\, t^2 - t^3$.

the adjusted saved value. We calculate the sum of the cubes of the consecutive numbers from one through seven to get 784. Multiply this by 27, which is the cube of three, to get 21168. Subtract from this the adjusted saved value, and what is left equals 14540, and that is the desired result.[87]

If the first number exceeds the common difference by a second given number, take the sum of the given number of consecutive numbers in the arithmetic sequence with the given common difference [and beginning with that common difference]. You already know the manner of doing this from what preceded. Multiply the result by three times the square of the second given number, and save the result. Then take the sum of the given number of squares of consecutive numbers in the [same] arithmetic sequence with the given common difference, and multiply the result by three times the second given number, and save the result. Finally, multiply the cube of the second given number by the first given number and add the result to the two saved values. The result is the first adjusted saved value. After this, take the sum of the given number of cubes of consecutive numbers of the [same] arithmetic sequence with the given common difference, and add it to the first adjusted saved number. This is the desired result.

As a model, consider our previous example, where my intention is that the first number exceeds the common difference by two. We know that the sum of seven consecutive squares beginning with three with a common difference of three is 1260. Multiply 1260 by three times the second given number, which is 6, resulting in 7560, and save this result. And also, the sum of seven consecutive numbers [beginning with three] and with an increase of three is 84. Multiply this by three times the square of the second given number, which is 12, to get 1008. And save this result too. Finally, multiply the cube of the second given number, which is 8, by the first given number, which is 7, to get 56. Add this to the two saved values to get 8624 which is the first adjusted value. Next, the sum of the cubes of the seven numbers with an increase of three is 21168. Add this to the adjusted first value to get 29792, and that is the desired result.[88]

This is true, because if we subtract 2 from each term, we form an arithmetic sequence with difference 3 [and beginning with 3]. The cube of each term is, however, less than the cube of the original increased term by the triple product of the square of the number with 2, increased by the triple product of the term with the square of 2 and by the cube of 2. If we add these to all the terms, we get what we have asserted and you will find this so. You can find the reason in the other case with a bit of thought.

. . .

If you want to add a given number of terms of a geometric sequence with given ratio, subtract the first from the second and then the ratio of this difference to the first term is equal to the ratio of the difference between the last and first terms to the sum of the

[87] $1^3 + 4^3 + 7^3 + 10^3 + 13^3 + 16^3 + 19^3 = \left(1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3\right) \times 3^3 - \left[\left(1^2 + 4^2 + 7^2 + 10^2 + 13^2 + 16^2 + 19^2\right) \times 3 \times 2 + (1 + 4 + 7 + 10 + 13 + 16 + 19) \times 3 \times 2^2 + 2^3 \times 7\right].$

[88] $5^3 + 8^3 + 11^3 + 14^3 + 17^3 + 20^3 + 23^3 = \left(3^3 + 6^3 + 9^3 + 12^3 + 15^3 + 18^3 + 21^3\right) + \left[\left(3^2 + 6^2 + 9^2 + 12^2 + 15^2 + 18^2 + 21^2\right) \times 3 \times 2 + (3 + 6 + 9 + 12 + 15 + 18 + 21) \times 3 \times 2^2 + 2^3 \times 7\right].$

entire sequence [up to the term before the last], as is proved at the end of the 9th book of Euclid. As an example, if you wanted to add six proportional numbers with ratio 3 and with first term 4, you already know that the second is 12, and the last is 972. Subtract the first, which is 4, from the second, leaving eight. The ratio of 4 to 8 is one half. Subtract 4 from the last leaving 968. Taking half gives 484, and adding to 972 gives 1456, which is the desired result.

Variations on the Rule of Three (from part II, chapter 6)

You already know that with every four proportional numbers, the product of the first with the fourth equals the product of the second with the third. As this is so, it should be clear to you when given some numbers, and given a second number that is a multiple of one of these numbers, how to extract the rest of the multiples so that the multiples are in the same previous given ratio. It is worth knowing that if you multiply one of the numbers by the given second number, and divide by the number of which it is a multiple, you get the multiple of the number that you multiplied by the second given number.

For example, given the numbers A, B, C, D, E, and G, a multiple of D, we want to find multiples of A, B, C, D, E. Multiply G by A, and divide by D, giving H, because the product of A and G equals the product of H and D. So the ratio of A to D is equal to the ratio of H to G, and by exchanging, the ratio of A to H is equal to the ratio of D to G. Similarly, it is clear that if you multiply B by G and divide by D, the result I is the multiple for B. So you can multiply C by G and divide by D to get J, the multiple for C. And, you can multiply E by G and divide by D, to get K, the multiple for E. So we have found the multiples of A, B, C, D, E, and they are H, I, J, G, K. It is clear that H, I, J, G, K are proportional to A, B, C, D, E, as was to be proved.

Also, if no number of the proportional numbers was known to us, but we did know the sum of two or three of them, then that is enough to get the numbers. For example, say you know in our last example that the sum of H, G, K equals M. We want to get the multiples of the given numbers A, B, C, D, E. Take the sum of the numbers of which H, G, K are multiples, namely A, D, E, and call this N. The ratio of N to M is equal to the ratio of A, B, C, D, E to their multiples. Accordingly, multiply M by A and divide by N, to get H. And in this way, we find the numbers I, J, G, K. I claim that the numbers H, I, J, G, K are the desired numbers.

. . .

The author writes: this completes the sixth chapter of this work, and with its completion, the book is complete. And the praise is to God alone. And its completion was on the first of the month of Nisan in the year 81 of the 6th millennium, when I reached the 33rd year of my years. And bless the Helper. [The year was 5081, i.e., 1321].

Problem section

Many of the problems that Levi brings are standard problems common to many of his contemporary and predecessor mathematical cultures (Arabic, Indian, Chinese, and Mesopotamian). Note, however, that Levi formulates his problems in abstract terms, and does not shy away from elaborate calculations with fractions.

6. A certain full container has various holes in it. One of the holes lets the contents of the container drain out in a given time; a second hole lets the contents drain in a second given time; and so on for each of the holes. All the holes are opened together. How much time will it take to empty the container?

First, calculate what drains from each hole in one hour and add the values all together. Note the ratio of this to the full container. This ratio equals the ratio of one hour to the time needed to empty the container.

For example, a barrel has various holes: the first hole empties the full barrel in 3 days; the second hole empties the full barrel in 5 days; another hole empties the full barrel in 20 hours; and another empties the full barrel in 12 hours. Therefore, the first hole empties one of 72 parts of the barrel in an hour; the second hole, one of 120 parts; the third hole, one of 20 parts; and the fourth hole, one of 12 parts. When we add them all up, the total that empties from all the holes in an hour is 56 of 360 parts of the full barrel. We divide 360 by 56, to get 6 whole and 25 firsts and 43 seconds. Therefore, the time to empty the barrel is approximately 6 hours, 25 firsts,[89] and 43 seconds. The reason for this is clear.

13. One man hires another to work a given number of days, for a fixed wage. This job requires the hiring of a certain number of men per day, each of whom leads a certain number of animals, each of which carries a given number of measures and walks a given distance. The hired man deviates from some or all of these numbers. How much should his wages be?

Take the product of all the values that were stipulated, and make a note of it. Furthermore, take the product of the actual values that were accomplished; and the ratio of the number you noted to this product equals the ratio of the wages he promised him to the wages he owes him.

For example, Reuven hired Simon to work 9 days for 10 *litra*. The job stipulated the hiring of 13 men each day, each of whom leads seven animals, each of which carries 15 measures and walks 6 *parsas*.[90] Simon provided 8 days, 17 men, each of whom led 6 animals, each of which carried 11 measures and walked 7 *parsas*. The product of the stipulated numbers, 9, 13, 7, 15, and 6 equals 73 thousand and 710, which is noted. The product of the accomplished numbers, 8, 17, 6, 11, and 7, equals 62 thousand and 832. The ratio of 10 *litra* to what he owes him equals the ratio of the noted value to 62 thousand and 832. If you multiply 10 *litra*, the first number, by the fourth number, which is 62 thousand and 832, and you divide the result by the noted value, you will get the number of *litras* and fractions of a *litra* that he owes him. This is 8 and one half *litra*, and a thousand and 7 hundred and 85 of 73 thousand and 710 parts of a *litra*, which is 5 *pashuts* and 59 thousand and 850 of 73 thousand and 710 parts of a *pashut*.

This is right, because the ratio of what he owes to what he stipulated equals the ratio of what he did to what he agreed to do. And the ratio of what he did to what he agreed to do is composed of the ratios of the numbers that were stipulated to the corresponding numbers that were accomplished. This composed ratio, as we already explained, equals

[89] Firsts refer here to minutes.

[90] *Litra* is a coin denomination which contains 20 *dinars*, each dinar containing 12 *pashuṭs*. (See footnote 11.) A *Parsa* is a distance unit.

the ratio of the product of the stipulated numbers to the product of the numbers that were accomplished. Use this as a model.

As noted above, problem 18 is an abstract version of the "men finding a purse" problem, a problem that seems to have appeared first in India (see Appendix 3), but is also found in the work of Leonardo of Pisa and other medieval European mathematicians. The original version of the problem was indeterminate, but Levi shows here how to find a definite answer if one is given one additional specific piece of information.

18. We add one number to a second number; and the ratio of the result to a third number is given. When we add the first number to the third number, the ratio of the result to the second number is a second given number. One of the three numbers is known. What is each of the remaining numbers?

You already know how to find three numbers that correctly meet these conditions, so extract them.[91] Since you know one of the numbers corresponding to one of the three, you can extract the other corresponding numbers, and that is what was requested.

For example, when you add the first number to the second number, its ratio to the third equals 3 wholes and 2 fifths and a seventh. When the first is added to the third, its ratio to the second equals 7 wholes and 2 thirds and a fourth. The second number is 30. We want to know: what is the value of each remaining number?

First of all, extract three numbers, using the procedure described in part one of this book. Accordingly, subtract one from the product of 3 wholes and 2 fifths and a seventh with 7 wholes and 2 thirds and a fourth. This leaves 27 wholes and a third of a seventh, which is the first number. Add one to 3 wholes and 2 fifths and a seventh, to get 4 wholes and 2 fifths and a seventh, which is the second number. Also, add one to 7 wholes and 2 thirds and a fourth, and the result you get is the third number, which is 8 wholes and 2 thirds and a fourth. You already know that the number corresponding to the second number is 30.

First	*Second*	*Third*
27 wholes and a third of a seventh	4 wholes and 2 fifths and a seventh	8 wholes and 2 thirds and a fourth
178 wholes and 98 of 159 parts of one	30	58 wholes and 281 of 318 parts of one

Thus the number corresponding to the first is 178 wholes and 98 of 159 parts of one; and the number corresponding to the third is 58 wholes and 281 of 318 parts of one. These three numbers are what were requested, so investigate and find them.

. . .

We will explain why these corresponding numbers are the desired ones. This is because the ratio of the first of the former numbers to the second of them equals the ratio of the first of the latter numbers to the second of them; and the ratio of the second

[91] See part I, problem 58 above.

of the former numbers to the third of them equals the ratio of the second of the latter numbers to the third of them. The compound of these ratios equals the ratio of the first of the former numbers to the third, which equals the ratio of the first of the latter numbers to the third. By adding these together, the ratio of the sum of the first and second former numbers to the third equals the ratio of the sum of the first and second latter numbers to the third. But in our example, the ratio of the sum of the first and second former numbers to the third is 3 wholes and 2 fifths and a seventh. Hence the ratio of the sum of the first and second latter numbers to the third is 3 wholes and 2 fifths and a seventh. Similarly, the ratio of the sum of the first and third former numbers to the second is 7 wholes and 2 thirds and a fourth. Use this as a model.

II. NUMEROLOGY, COMBINATORICS, AND NUMBER THEORY

Numbers were not only treated as mathematical entities—they were also given natural and mystical significance. An excerpt from Ibn Ezra's *Book of One* shows how the different approaches intermingled in scholarship. We follow with two discussions of combinatorics. In the first, Ibn Ezra calculates the number of possible conjunctions of a given number of planets from among the seven planets. In the second, Levi ben Gershon engages in an abstract and general discussion of permutations and combinations. We proceed with Levi's elegant treatment of harmonic numbers, showing that different products of powers of two and three cannot have difference one, except for the four known harmonic couples. We end with a discussion of amicable numbers in the Hebrew literature.

1. ABRAHAM IBN EZRA, *SEFER HA'EḤAD* (*THE BOOK OF ONE*)

This book by Abraham ibn Ezra (see section I-1) summarizes knowledge about numbers from various mathematical disciplines, along with other natural sciences. It also has clear mystical overtones and is written in a terse style that is sometimes hard to fathom.[1] This translation covers the treatment of the number 4.

Four is the first visible square.[2] It is the first even-even [of the form $2n$ with n even], and it is the first composite [non-prime number]. Indeed, the first decimal order [*maʿarekhet*] consists of nine [digits], wherein one is the foundation of counting, leaving four primes: 2, 3, 5, and 7, and the other four composite. Its sum with the numbers that precede it [$1 + 2 + 3 + 4$] is ten, which is the beginning of the analogous multiple [*klal*, the next decimal order].[3] Four is their root [the base of the triangular number 10].

Since numbers unto a prime are like unto an indivisible unit, it is opposite; such are 2 and 5, 3 and 6, 4 and 7, and so on without end.[4] Therefore, the fourth astrological sign is the opposite of the first; for heat and cold are active [properties of the four elements],

[1] For the various meanings of numbers in Ibn Ezra's work, see [Langermann and Simonson, 2000].

[2] Although one is also a square, it is not so "visibly," since its root is equal to itself.

[3] Compare the opening of *The Book of Number* (section I-1).

[4] This is a possibly corrupt, baffling statement. It might suggest that integers which, like 1 and 4, are three apart, are opposite to each other in that one is prime and the other is not. But this would work only as far as 5 and 8. Another interpretation is that such numbers are opposite in that one is even and the other odd. The next sentences validate this opposition from an astrological point of view.

whereas the remaining two, wet and dry, are passive.[5] Every fourth sign is the opposite of the first in the active [property], but the fiery signs alone are the opposite both in active and passive properties].[6] For this reason the astrologers say that the quartile aspect [pairs of signs that are three apart] is enmity.

A sextile aspect [pairs of signs that are two apart] is half-friendship, for the third sign is identical in the active property of the element to the first, but opposite in the passive; therefore, they say that it is half-friendship. Trine [pairs of signs that are four apart] is complete friendship, for at the trine aspect is the fifth sign, which has the same element as the first. Therefore, it is the aspect of complete friendship in the active and the passive. And so 1 and 5, since both preserve themselves;[7] likewise 2 and 6, since both are even-odd [of the form *2n* with n odd]; and so also 3 and 7, since neither are even, and their components are similar.[8] Not so, however, are 1 and 3, 2 and 4, and 3 and 5.[9]

Opposition [the aspect of half the orb, where signs are six apart] (and half-wise the quartile aspect [where signs are 3 apart]), are aspects of enmity in the passive property.[10] Do not be puzzled that the seventh house [which is in opposition to the first] indicates one's mate [*ʿezer*, woman]; as the author of *Sefer Yeṣira* said: A.M.Sh. for male, and A.Sh.M. for female.[11]

Now the orb, which is one, is divided by its diameter. If the diameter too is cut [in half], the aspect is quartile. If you place a point at the fourth [of the diameter and construct the perpendicular chord], the orb will be divided into three equal sections, forming a triangle. Its half is one-sixth of the orb. No other number can divide it [the circle] except in thought or with fractions.[12]

Four is the beginning of non-equilateral acute triangles. All subsequent consecutive numbers follows its rule.[13] Now in the first obtuse triangle [sides 2, 3, 4] the [square on

[5]The signs are divided as follows: Aries, Leo, and Sagittarius are fire signs (hot and dry); Taurus, Virgo, and Capricorn are earth signs (cold and dry); Gemini, Libra, and Aquarius are air signs (hot and wet); and Cancer, Scorpio, and Pisces are water signs (cold and wet).

[6]A pair of signs that are three apart (like 1 = Libra and 4 = Aries) is always in opposition with respect to the active property of its element (hot/cold). A pair that begins with a fire or air sign is opposite also with respect to the passive property (dry/wet). The text should probably have "fire and air" rather than "fire."

[7]That is to say, 1, 5, and other numbers having either of these as their final digit will preserve their final digit in all higher powers.

[8]According to Comtino, they are similar insofar as both are sums of consecutive odd and even numbers: $3 = 1 + 2$, and $7 = 3 + 4$.

[9]Comtino suggests that this means that we cannot find any full and satisfactory similarity between these pairs, such as were found for the pairs listed above.

[10]Opposition is the aspect of enmity in the passive, whereas the quartile aspect is an aspect of enmity in the passive only for pairs starting with fire or air signs.

[11]The first house is associated with masculinity, and its opposite, the seventh, with femininity. This is also the case with the permutations of the letters A.M.Sh in *Sefer Yesira*: the first is associated with masculinity, and its opposite among the six possible permutations, which is the fourth permutation A.Sh.M., is associated with femininity. *Sefer Yesira* (the *Book of Creation*; see [Kaplan, 1995]) is a mystical treatise concerning the creative power of letter and number combinations.

[12]The text differs slightly among versions and suggests different readings. In this reading it is claimed that 12 can't be divided without remainder except by 2, 3, 4, and 6. According to another reading, the point is to show that divisions by four (first of the circle, then of the diameter) can produce the quartile, trine, and sextile aspects.

[13]In the previous sections, the triangles with sides 2, 3, 4 (obtuse) and 3, 4, 5 (right angle) were discussed. The triangle with sides 4, 5, 6 is acute, and so are all the following triangles with consecutive side lengths.

the] longest side exceeds the [sum of the squares on the other] two by the numerical value of the middle [$4^2 - (2^2 + 3^2) = 3$], but in the acute [triangle] the longest side falls short by the numerical value of the middle [that is, in the triangle 4, 5, 6 we have: $4^2 + 5^2 - 6^2 = 5$]. I will now give you a rule. Given three consecutive numbers, the smallest of which is at least four, and you wish to know by how much the [sum of the squares of] the two [smaller sides] exceeds the [square of] the longest, always subtract four from the middle and multiply what remains by the middle. For example, 10, 11, 12. We subtract 4 from 11, leaving 7; we multiply it by 11, producing 77. It is the deficiency of the square on the longest side [with respect to the squares on the two smaller sides; $10^2 + 11^2 - 12^2 = 77$].

Know that some angles [of triangles whose sides are integer triplets] are very wide; others are close to being right-angles, for example, 4, 8, 9. Likewise there are acute [angles] which are half of a right angle, or a third or as little as one degree. But do not think that you can form a triangle out of any [set of three] number[s] that you want. For an acute [triangle with consecutive side lengths] cannot have [a side] which is less than 4. Nor can one side of a triangle be greater than [the sum] of the two [other sides]. Nor can there be a right triangle in which the numerical values of the sides are distant [from each other]. They are either one set [of consecutive numbers], such as 3, 4, 5 and their multiples; or the two smaller sides are consecutive, for example, 20, 21, 29; or the greater ones are consecutive, as in 5, 12, 13.[14] There are very obtuse triangles, such as 10, 17, 26, and slightly [obtuse], such as 10, 23, 26. Look at 10, 24, 26 [which has a right angle]; but 10, 25, 26 is slightly acute. Hence we can form neither a right nor an acute [triangle] in which the sides are distant.

I shall now give you a rule. Know that by reckoning in the first order [i.e., numbers 1–9], there are 84 triangles with no two equal sides, but only 33 of them are true. Likewise, not all [possible] isosceles triangles are true.

Four is the first non-prime [*sheni*, literally, “secondary”].[15] Therefore even numbers are always non-prime, that is, composite, except for the number 2, due to the power of the one, which is its main influence. Every number multiplied by 1 will not increase [1 times *n* is *n*], since it [1] is the essence of every number. Therefore, the first square is 4. Every square multiplied by a square is a square, and divided by a square, is a square; the ratio of a square to a square is also a square. Therefore, the measures of all of the scholars are in squares.[16]

2. ABRAHAM IBN EZRA, *SEFER HAʾOLAM* (*BOOK OF THE WORLD*)

Sefer Haʿolam (*Book of the World*) by Ibn Ezra discusses the meaning of celestial conjunctions and aspects. It opens by counting all possible conjunctions of the seven known planets, demonstrating some systematic combinatorial reasoning. To calculate the number of different sets of *n* elements out of 7 planets, a recursive method is used, taking partial sums

[14] Since there obviously are right triangles where the sides are not multiples of triplets with a consecutive pair (e.g., 12, 35, 37), the meaning of “distant” here and in the final sentence of this paragraph is not quite clear.

[15] At the beginning of this text, Ibn Ezra had used the term *murkav,* which we translate as “composite,” rather than *sheni* (secondary) used here. He thus transmits both Greek terms, *deuteros* and *sunthetos.*

[16] This is yet another baffling or corrupt sentence. It might refer to the measurements of areas.

of the sequence 1,2, . . . ,7, then taking partial sums of the sequence of these partial sums, and so forth. In this text Ibn Ezra does not mention the symmetry between the combinations of k elements out of 7 and of $7 - k$ elements of 7, but he does note this symmetry and uses it to simplify calculations in his later long commentary to Exodus 33:21 [Ibn Ezra, 1991]. This excerpt was previously translated and analyzed in [Ginsburg and Smith, 1922].

If you come across Abu Maʿashar's[17] *Book on the Conjunctions*[18] *of the Planets*, you would neither like it nor trust it, because he relies on the mean motion for the planetary conjunctions. No scholar concurs with him, because the truth is that the conjunctions should be reckoned with respect to the zodiac. Nor should you trust the planetary conjunctions calculated according to the [astronomical] tables of the Indian scholars, because they are wholly incorrect. Rather, the correct approach is to rely on the tables of the scholars of every generation who rely on experience.

There are 120 conjunctions [of the seven planets].[19] You can calculate their number in the following manner: it is known that you can calculate the number that is the sum [of all the whole numbers] from one to any other number you wish by multiplying this number by [the sum of] half its value plus one-half. As an illustration, [suppose] we want to find the sum [of all the whole numbers] from 1 to 20. We multiply 20 by [the sum of] half its value, which is 10, plus one-half, and this yields the number 210.

We begin by finding the number of double conjunctions, meaning the combinations of only two planets. It is known that there are seven planets. Thus Saturn has 6 [double] conjunctions with the other planets. [Jupiter has 5 double conjunctions with its lower planets, Mars has 4, and so on. So we need to add the numbers from 1 to 6]. Hence we multiply 6 by [the sum of] half its value plus one-half, and the result is 21, and this is the number of double conjunctions.

We want to find the [number of] triple conjunctions. We begin by taking Jupiter and Saturn, and [then take] any of the other five [planets] with them; the result is the number 5. [Then we move on to conjunctions composed of Saturn, Mars and with one of the lower four, then Saturn and the Sun with one of the lower three and so on. Altogether], we multiply it [5] by 3, which is [the sum of] half its value plus one-half, and the result is 15, and those are Saturn's [ternary] conjunctions. Jupiter should have 4 [triple] conjunctions [with Mars and the lower planets; continuing by the same method], we multiply that [4] by [the sum of] 2 plus one-half, and the result is 10. Mars has 3 conjunctions; we multiply them by 2, and the result is 6. The Sun has 2 conjunctions; we multiply them by the sum of 1 plus one-half, and the result is three. Venus has one conjunction with the planets beneath it. So the total is 35, and this is the number of triple conjunctions.[20]

[17] Abu Maʿashar (787–886) is the most prominent astrologer of the Middle Ages. He formulated the standard expression of Arabic astrology in its various branches, creating a synthesis of the Indian, Persian, Greek, and Harranian theories current in his days. Despite the critique presented here, he is also Ibn Ezra's most important Arabic astrological source.

[18] The conjunctions (*Maḥbarot*) are astronomical events where several planets appear close together in the sky. These events are thought to have substantial astrological impact.

[19] The seven planets, in order from the lowest to the highest, are the moon, Mercury, Venus, the sun, Mars, Jupiter, and Saturn.

[20] In anachronistic terms, Ibn Ezra is calculating that $C_{7,3} = C_{6,2} + C_{5,2} + C_{4,2} + C_{3,2} + C_{2,2}$, where each $C_{k,2}$ is shown to be the sum of integers up to $k - 1$.

We wish to find out the quadruple conjunctions. We begin with Saturn and Jupiter, and Mars with it. For [these] three [planets] to conjoin [with one of the four lower planets], we start with 4 conjunctions. We multiply them by 2 and one-half and the result is 10, [namely, ten quaternary conjunctions that begin with Saturn and Jupiter]. Then come the [quadruple] conjunctions of Saturn and Jupiter [should be Mars] with the others [lower planets], and we start with 3 [conjunctions]. We multiply that by 2 and the result is 6 [quadruple conjunctions beginning with Saturn and Mars, skipping Jupiter], and [the partial sum] is sixteen. Then we have Saturn with Mars [should be the sun], and there are 2 [quadruple conjunctions]. We multiply them by 1 and one-half and the result is 3. Then comes another conjunction [Saturn, Venus, Mercury and the moon], and [the total] for Saturn is 20 [quadruple] conjunctions. Now we have Jupiter with 3 [conjunctions]. We multiply that by 2 and the result is 6. Then come two [conjunctions]. We multiply that by 1 and one-half and the result is 3. Then comes one conjunction. [The total of] Jupiter's [quadruple] conjunctions is 10. Then we have Mars with two [conjunctions]. We multiply that by 1 and one-half and the result is 3. Then comes one conjunction, making 4 [quadruple] conjunctions [beginning with Mars]. The Sun has one [quadruple] conjunction with the planets beneath it. So the sum total is 35 quadruple conjunctions.[21]

We wish to find the quintuple conjunctions. We find 15 for [those beginning with] Saturn, 5 for Jupiter, and 1 for Mars. So there are 21 quintuple conjunctions. As for sextuple conjunctions, there are 6 for Saturn and 1 for Jupiter, making a total of 7. There is one septuple conjunction. So we have obtained 120 conjunctions. All these conjunctions are odd numbers that are divisible by seven [except the last].

3. LEVI BEN GERSHON, *MAʾASE ḤOSHEV*

Unlike Ibn Ezra's, Levi's combinatorics from *Maʾase Ḥoshev* (see section I-6) are abstracted from any practical context and include accurate proofs. In part I, Levi proves some of his results on permutations and combinations by using a form of mathematical induction; he presents the inductive step before stating the theorem and writing the initial step.[22] Then, in part II, Levi uses the theoretical results to do calculations.

In what follows, we use the modern words "permutations" and "combinations" for Levi's verbose formulations. Instead of "arrangements of a certain number of distinct terms from another (larger) number of distinct terms that are exchanged either by order or by terms," we write "permutations of n elements out of a set of m elements" or $P_{m,n}$. Instead of "arrangements of a certain number of distinct terms from another (larger) number of distinct terms that are exchanged by terms," we write "combinations of n elements out of a set of m elements" or $C_{m,n}$. And instead of "arrangements of a certain number of distinct terms that are exchanged only by order," we write "permutations of n elements" or P_n.

From part I

63. If the number of permutations of a given number of different elements is equal to a given number, then the number of permutations of a set of different elements containing

[21] In this paragraph, Ibn Ezra calculates quadruple conjunctions according to the anachronistic formula $C_{7,4} = C_{6,3} + C_{5,3} + C_{4,3} + C_{3,3}$, and then uses the above procedure to calculate $C_{k,3}$.

[22] See [Rabinovitch, 1970] for further discussion of mathematical induction in Levi's work.

one more element equals the product of the former number of permutations with the number after the given number.[23]

Let the terms be *A*, *B*, *C*, *D*, *E* and their number be *G*. Let the number following *G* be *H*, and let the number of ways to arrange the terms *A*, *B*, *C*, *D*, *E* be *I*. And let the terms *A*, *B*, *C*, *D*, *E*, *F* add one term to the terms *A*, *B*, *C*, *D*, *E*, and thus their number is *H*. We say that the number of permutations of the terms *A*, *B*, *C*, *D*, *E*, *F* is the product of *I* with *H*.

The proof is that if you place *F* first followed by each of the permutations of *A*, *B*, *C*, *D*, *E*, the [new] permutations will remain distinct, and therefore the number of permutations where *F* is the first term equals *I*. Similarly, since the number of permutations of *A*, *B*, *C*, *D*, *E* equals *I*, therefore the number of permutations of *A*, *B*, *C*, *D*, *F* is also *I*. And when you place *E* first followed by all these permutations, you are left with distinct permutations, and thus the number of permutations where *E* is first equals *I*. And in this way it is explained that when each one of the terms is placed first, the number of permutations is *I*. If so, then the total of these permutations is *I* multiplied by the number of terms. However, there are *H* terms. Therefore, the number of permutations of *A*, *B*, *C*, *D*, *E*, *F* is the product of *H* with *I*.

It is clear that among all those permutations counted, there are no two identical ones, because when a certain element is first, there are no identical permutations because the permutations before attaching it were distinct and they remain distinct when it is attached to them. And there is no doubt about that when the first elements are different. This being the case, it is clear that among those permutations that we have counted, there are no two identical ones.

We also say that there are no permutations besides those. For if there were, let *DFECAB* be such a permutation. But in this case *D*would have been added to the remaining elements *FECAB*, so *DFECAB* is one of the permutations we have counted. And since there are no two identical permutations and there are no permutations except those, it follows that the number of permutations of *A*, *B*, *C*, *D*, *E*, *F* is the product of *H* with *I*, which is what we wanted to prove.

And thus, it can be understood that the number of permutations of a given number of terms is the number built from consecutive numbers starting with one and their number is the number of these terms.[24] The number of permutations of 2 is 2, and that is built from the numbers 1 and 2. And the number of permutations of 3 is the product of 3 with 2, and this equals the number built from 1, 2, 3. And similarly, this can be shown without limit.

64. The number of permutations of two terms from a given number of distinct terms is equal to the product of the given number and the number that precedes it.[25]

65. When you are given a number of terms and the number of permutations of a second given number from these terms is a third given number, then the number of permutations of the number following the second given number from these terms is the product of the given third number by the excess of the first given number over the second number.[26]

[23] $P_{n+1} = (n+1)\,P_n$. The proof ends with an inductive style point about $n!$.
[24] $P_n = n!$.
[25] $P_{n,2} = n\,(n-1)$.
[26] $P_{H,I+1} = P_{H,I}\,(H-I)$, where, in Levi's lettering, $L = P_{H,I}$.

Let the terms be *A*, *B*, *C*, *D*, *E*, *F*, *G*, and let their number be *H*. Let *I* be different from *H* and less than *H*. Let the number of permutations of *I* elements from these elements be equal to *L*; let *M* be the number following *I*. Let the difference between *H* and *I* be *N*. I say that the number of permutations of *M* elements from these terms is equal to the product of*L* and *N*. Let one of the permutations of *I*elements be *ABC*; then the remaining elements are *D*, *E*, *F*, *G* and their number is equal to *N*. Putting each one of the elements *D*, *E*, *F*, *G* together with the permutation *ABC*, distinct permutations result and the number of elements in such a permutation is *M*, because one more element has been added. Since the number of *D*, *E*, *F*, *G* is *N*, the new permutations stemming from *ABC* will number *N*, and it is clear that the number of new permutations stemming from each permutation of *I* elements, distinct in their order or in their elements, is *N*. So the total number of these permutations of *M* elements is the number *N* multiplied by the number of permutations of *I* of these elements. But the number of permutations of *I* of these elements is *L*; thus the number of permutations of *M* of these elements is the product of *N* by *L*.

We say that among all these permutations that we have counted, no two are identical permutations. Indeed, to one permutation, distinct elements have been joined, from one time to the next, and from this it follows that these permutations are distinct, and there is no doubt that distinct permutations will not become identical when any element is joined to them. Thus there are no two identical permutations.

We say that there are no permutations besides those that we have counted. For if it were possible, let one such permutation be *FDBG*. However, the permutation *DBG* has been joined with each one of the remaining elements at the first place, and one of these elements is *F*. Thus the permutation *FDBG* is one of the permutations that we counted. There being no two identical permutations among those that we counted and there being no permutations besides these, the number of permutations of *M* of these elements is the product of *N* by *L*, which we wanted to prove.

And so it is clear that the permutations of a given number from a second given number of terms is equal to the number built from consecutive numbers. Their number is equal to the first given number, and the last one is the second given number. Let the number of terms be 7, and the consecutive numbers starting from 1 are 1, 2, 3, 4, 5, 6, 7. It is clear that the number of arrangements of two of these is the product of 6 with 7—the number of numbers is 2; they are consecutive, and the last of them is 7. The number of permutations of 3 of them is equal to the product of 5 with the product of 6 and 7, because the excess of 7 over 2 is 5. This equals the product built from 5, 6, 7. Also, these numbers are three numbers; they are consecutive, and the last of them is 7. Similarly, it can be explained with any number you like.[27]

66. When there is a given number of terms, and the number of combinations of a second given number from these terms is a third given number, and the number of permutations of as many terms as this second given number is a fourth given number, then the number of permutations of the second given number from as many terms as the first given number is equal to the product of the third given number by the fourth given number.[28]

[27] In modern notation, Levi has proved that $P_{m,n} = (m - n + 1) \ldots (m - 2)\,(m - 1)\,m$.

[28] Using Levi's lettering, this result, in modern notation, is $P_{G,H} = P_H C_{G,H}$.

Let the elements be *A*, *B*, *C*, *D*, *E*, *F* and let their number be *G*. Let the number of combinations of *H* elements out of *G* be *J* and the number of permutations of *H* elements be *L*. I say that the number of permutations of *H* elements out of *G* elements is equal to the product of *J*and *L*. Let one of the ordered sets from the combinations of *H* elements be *BCD*; then all the permutations of this set produce*L* ordered sets. In the same way, one shows that for each of the ordered sets from combinations of *H* terms out of all the terms one can form *L* ordered sets. So out of the total of all these sets, one obtains ordered sets by multiplying the number by *L*. Since the number of these sets is *J*, the number of the ordered sets is the product of *J* and *L*.

We claim now that among all the chosen ordered sets, no two are the same. For where the elements are different, they are permuted, and the number of permutations is *L*, as we had assumed. Without a doubt, moreover, two ordered sets containing different elements cannot become equal through permutations.

We claim further, that there are no other ordered sets besides the ones we have already counted. For suppose this were possible, say, by the ordered set *FDB*. But all the elements *BDF* are already permuted and one of the permutations is *FDB*. But *FDB* is one of the ones already counted, so there is no further ordered set available. So since, among the counted ordered sets no two are equal and no further ones are available, therefore the number of permutations of *H* elements out of the elements *A*, *B*, *C*, *D*, *E*, *F* is equal to the product of *J* and *L*.

67. When there is a given number of distinct terms, and the number of permutations of a second given number from these terms is a third given number, and the number of permutations of the second given number of terms is a fourth given number, then the number of combinations of the second given number from the given number of terms equals the number of units by which the fourth number counts the third number.[29]

68. When there is a given number of distinct terms, and the number of combinations of a second given number from these terms is a third given number, and the excess of the first given number over the second given number is a fourth given number, then the number of combinations of the fourth given number from those terms is equal to the third given number.[30]

From chapter 4 of part II

If you wish to find the number of permutations of a second given number of elements out of a first given number of distinct elements, you should know that the number of permutations of two elements is equal to the product of the first given number and the previous number and that the number of permutations of three elements has a ratio to that of two elements as the ratio of the difference between the first number and 2 to 1. Furthermore, the number of permutations of four elements has a ratio to that of three elements as the ratio of the difference of the first number and 3 to 1, and so on without end.

Therefore, you proceed as follows: Form the number from the product of as many numbers following each other as the second given number, with the last being equal to

[29] $C_{m,n} = P_{m,n}/P_n$.
[30] $C_{m,m-n} = C_{m,n}$.

the first given number. The result is what one desires. As an example, you want to know the number of permutations of five elements out of eight. Since the second given number is 5, take the product of five consecutive numbers so that the last one is 8, that is, the product of 4, 5, 6, 7, 8. This product is 6720, which is the number of permutations of five elements out of eight. This is so, because the number of permutations of two elements is the product of 7 and 8 and of three elements is the product of 6 by the product of 7 and 8, as was previously shown, and the number of permutations of four elements is the product of 5 by the product of 6, 7, and 8 and of five elements is the product of 4 by the product of 5, 6, 7, and 8, and so forth. This is clear from the above.

If you wish to find the number of combinations of a second given number of elements out of a first given number of different elements, find the number of permutations of the second given number out of the first given number of different elements and note that. Then find the number of permutations of the second number. As many times as that number divides the number first found is what you are seeking. As an example, suppose you want to know the number of combinations of five elements out of eight. Find how often the product of 1, 2, 3, 4, 5 divides the product of 4, 5, 6, 7, 8. The product of 4, 5, 6, 7, 8 is 6720 and the product of 1, 2, 3, 4, 5 is 120. Now 6720 contains 120 56 times, so 56 is the desired result. . . .

To make things easy for you, you should also know that the number of combinations of five elements out of eight elements is equal to the number of combinations of three elements out of these elements. In fact, this number is the number of times the product of 6, 7, 8, that is, 336, contains the product of 1, 2, 3, that is, 6. This result is 56, so the number of combinations of three elements out of eight elements is the same as that of five elements. The reason for this was presented earlier.

4. LEVI BEN GERSHON, *ON HARMONIC NUMBERS*

According to Levi ben Gershon himself, *On Harmonic Numbers* was commissioned by Phillipe de Vitry, a French composer and music theorist and also an official at the court of Philip VI, to answer a question about numbers formed by powers of 2 and 3. This question was perhaps related to the question of possible ratios giving harmonic tones. Recall that the ratio 2:1 gives an octave, 3:2 a fifth, 4:3 a fourth, and the difference between a fifth and a fourth (the quotient of the ratios) gives 9:8, a single tone. All these ratios are of the form $(n + 1) : n$, so the question was asked whether there are any other ratios of that form that can come from the basic ratios by composition. Levi's negative answer is based on an ingenious parity analysis of the various numbers involved.

Although Levi wrote the work in Hebrew, it was immediately translated into Latin (perhaps by Petrus of Alexandria), and the Hebrew version has been lost.

In the year 1342 of the incarnation of Christ, our work on mathematics having been completed, I was requested by the noted master of the science of music, Master Philippe de Vitry of the French kingdom, to demonstrate a certain hypothesis postulated in that science: all pairs of harmonic numbers differ by a number, except for 1 and 2, 2 and 3, 3 and 4, [and] 8 and 9.[31]

[31] Note that 1 is not considered to be a number; 2 is the smallest number.

A harmonic number is described as follows: A number is harmonic if, except for the unit, it is divisible only by 2 or 3, and the factors also are similarly divisible down to unity. Some examples are 1, 2, 4, 8, ... and 1, 3, 9, 27, ... , and also 6, 12, 18, 24, ... And I wish to satisfy him and demonstrate this principle in this place in our book. But since all such harmonic numbers are either, first, in continuous proportion with ratio 2, or secondly, in continuous proportion with ratio 3, or thirdly, of the type produced by multiplying a number of the first type by a number of the second, I will demonstrate the theorem on these proportions stated in other terms, as follows:

Of all the numbers successively proportional with ratio 2, and of all the numbers successively proportional with ratio 3, and of their mutual products, any two of these differ from each other by a number, except for: 1 and 2, 2 and 3, 3 and 4, and 8 and 9. If this were not true, there would be another pair of numbers defined as above, consisting of equal numbers or differing just by unity. But this conclusion is false. It appears thus that the initial proposition is true. I will demonstrate in what follows the falseness of the stated conclusion.

To facilitate the comprehension of this demonstration, I propose certain definitions. First, the numbers of the first class will be called "numerical powers of 2"; then the numbers of the second class will be called "numerical powers of 3." Those of the third class will be called "products of a numerical power of 2 by a numerical power of 3." Also, we will call the unit, the "element of the first rank in the first class and the second class," and the first number that follows, we will call the "element of the second rank," and the next number, the "element of the third rank," and so forth, to infinity. Finally, the second, fourth, sixth, eighth ranks and so on, we will call even ranks, the others, odd ranks. This concludes the definitions.

1. Every numerical power of 2 is even, because it is the product of an even number by the unit or by an even number, such as 2, 4, 8, 16, and so on.

2. Every numerical power of 3 is odd, because it is the product of an odd number by the unit or by an odd number, such as 3, 9, 27.

3. Every number that is the product of a numerical power of 2 by a numerical power of 3 is even, because it is the product of an even number by an odd number, such as 6, 12, 18.

4. Every numerical power of 2 is only divisible by another number that is a numerical power of 2, and every number of this same class is divisible by every number of lower rank in this same class and only gives a quotient of this same class. Similarly, every numerical power of 3 is only divisible by a number that is also a numerical power of 3.

This is true after Euclid [IX.11]. It follows that no numerical power of 2 is divisible by an odd number [IX.13].

5. The half of a numerical power of 3 less a unit is equal to the sum of the numerical powers of 3 of rank strictly less including the unit.

This is true by Euclid, IX.38 [IX.35], as one can verify.

6. Every numerical power of 2 and of 3 of odd rank is a square.

This is necessarily true by Euclid [IX. 8].

7. The sum of two numerical powers of 3 is even.

Let the two given numbers be AB and BC. If we subtract the unit BD from the number BC, there remains the number DC which is even. Since the number AD is even, it follows that the number AC is even, which was what we wanted to prove. And it has therefore been established that if one adds an even number of numerical powers of three, their sum will be an even number.

8. The sum of an odd number of numerical powers of 3 is odd.

Let the given numbers be A, B, C, D. Since the sum of A, B, C, D is even according to [7], if we add an odd number E, the total will be an odd number, which is what we wanted to prove.

9. The sum of a numerical power of 3 of odd rank and a number of the same class of the immediately following rank will be a square number divisible by 4.

Let A be the number of odd rank to which one adds B, the number of the following rank. Since A is a square by [6], and B is triple of A, that is, A multiplied by 3 is B, it follows that $A + B$[32] is four times A, which is a square number. And since 4 is also a square by [6], it follows that the ratio of A to $A + B$ is equal to the ratio of a square number to a square number. But A is a square; thus $A + B$ is a square and since $A + B$ is four times A, it follows that $A + B$ is divisible by 4. Thus the sum of a numerical power of 3 of odd rank and the number of the same class of the immediately following rank is a square number divisible by 4.

From what precedes, it follows that if two successive numerical powers of 3 are added, their sum is divisible by 4, because this sum is equal to four times the smaller of the two numbers. And it follows that if one adds an even number of consecutive numerical powers of 3, the sum is divisible by 4, because the sum of any two of these successive numbers is divisible by 4, after [9], and thus their total is divisible by 4.

10. The sum of any four powers of 3 of consecutive ranks is divisible by 5 and by 8; furthermore, the sum of these same consecutive numbers of the same class is divisible by 4.

Let A, B, C, D be the powers of 3. As 1 is to 9, the third element of this class, so is A to C and B to D. It follows that the ratio of $1 + 9$ to 1 is the same as the ratio of $A + C$ to A and $B + D$ to B. Therefore, $A + C$ is ten times A and $B + D$ is ten times B. It follows that $A + B + C + D$ is ten times $A + B$. But by [9], $A + B$ is four times A [and $C + D$ is four times C]. Thus $A + B + C + D$ is four times $A + C$ and therefore forty times A. But 40 is divisible by 5 and 8, and so $A + B + C + D$ is divisible by 5 and 8, and it is at the same time the sum of four consecutive powers of 3. And the sum of these same consecutive elements of this class, as we know, is divisible by 4, which is what we wanted to prove.

11. The fourth part of a number divisible by 8 is even.

Since, in fact, it is divisible by 8, it has an eighth part, which part, doubled, is the quarter. But this doubled number may be divided into two equal parts, and such a number is even. It follows that the fourth part of the given number is even, which is what we wanted to prove.

12. No even number is equal to an odd number or to the unit, because it differs by at least a unit from any of these.

13. No even number is equal to another even number, because it differs from any other by at least two.

[32] In the Latin manuscript, the sum of two or more quantities is indicated by placing the letters next to one another, separated by a dot. To make the reading easier, we have used the plus sign in what follows.

14. No odd number is equal to another odd number or to the unit, because it differs from any other by at least two.

These things being demonstrated, I will prove that the negation of the stated consequence is true, and therefore, first, that no two numbers of the defined classes are equal.

In propositions 15 through 20, Levi shows that no two numbers of the same class are equal, that a numerical power of 2 cannot equal a numerical power of 3, nor can a numerical power of 2 or 3 equal a number that is a product of two such numbers.

This decomposition and this mutual comparison of the numbers in question show that the negation of the first conclusion is true, namely that no two harmonic numbers are equal, and that is what we wanted to prove.

It remains to prove the negation of the second conclusion, namely, that no two numbers of the given classes differ by just a unit, with the exception of those that have been indicated above.

In propositions 21 and 22, Levi shows that no power of 2 (except for 1 and 2) can differ by a unit from another such power, that no power of 3 can differ by a unit from another power of 3, and that no product of a power of 2 by a power of 3 can differ by only a unit from another such product, because the parity of all numbers in each class is the same, so any two must differ by at least two.

23. No numerical power of 2 can differ by only a unit from a product of a numerical power of 2 by a numerical power of 3.

Let *A* be a numerical power of 2 and *B* the product of a numerical power of 2 by a numerical power of 3. So *B* is the product of *D*, a numerical power of 2 and *E*, a numerical power of 3. If *D* is greater than or equal to *A*, then the proposition is true. So suppose that the number *A* is greater than the number *D*. It is necessary by [4] that the quotient of *A* by *D* is a numerical power of 2. Let us denote that quotient by *F*. So *D* times *E* is *B*, and *D* times *F* is *A*. But *F* cannot be equal to *E*, because they differ by at least a unit according to [12], since one is even and the other is odd. It follows that the number *B*, which is the product of *D* and *E*, is not equal to the number *A*, which is the product of *D* and *F*. Even more, these differ by a number equal to the product of *D* by the difference between *E* and *F*. The resulting number is not less than the number *D* because the said difference is not less than 1. But the number *D* is at least equal to 2, since it belongs to the class of powers of 2. Thus the number obtained is at least equal to 2, so the number *A* differs from the number *B* by more than a unit, which is what we wanted to prove.

24. No numerical power of 3 can differ by only a unit from a number that is the product of a numerical power of 2 and a numerical power of 3.

The proof is similar to the proof of proposition 23 and is omitted.

25. With the exception of the second term of the class of powers of 2 and the second term of the class of powers of 3, the second term of the class of powers of 3 and the third of the class of powers of 2, and the third of the class of powers of 3 and the fourth of the class of powers of 2, no other numerical power of 2 differs by only a unit from any numerical power of 3.

If this were not true, then either there would be a particular numerical power of 3 that is greater by the unit than a particular numerical power of 2, and therefore when the unit is subtracted from this numerical power of 3 the remainder is equal to the numerical power of 2, but this is false. Or there would be a particular numerical power of 2 that is greater by the unit than a particular numerical power of 3, and therefore when the unit is added to the numerical power of 3, the sum is equal to the numerical power of 2, and this is false. The truth of these negations of the stated propositions will be demonstrated in a clear and evident fashion by the theorems that follow, the first of which is the following.

26. If the unit is subtracted from any numerical power of 3 of even rank, the remainder is not a numerical power of 2.

Let *AB* be such a numerical power of 3, an odd number by [2], from which one subtracts the unit *DB*. There remains the even number *AD*. If this is divided in half at the point *C*, then, by [5], the number *AC* is equal to the sum of all the terms of the class of powers of 3 preceding *AB*. This sum is odd, by [8], because the number of terms preceding *AB* is odd. It follows that the number *AD* is divisible by an odd number. It follows by [4] that *AD* is not a numerical power of 2, and that is what we wanted to prove. We are supposing here that *AC* is not the unit, which will be the case if *AB* is the second term of the class of powers of 3. But this is excluded by the hypothesis, for then *AD* will necessarily be a numerical power of 2, since *AC* will in fact be nothing else than the unit, which thus divides every number and whose double is the second term of the class of powers of 2.

27. If the unit is subtracted from any numerical power of 3 of any odd rank immediately following 4 or any multiple of 4, such as the ranks 5, 9, 13, 17, and so on, the remainder is not a numerical power of 2.

Let *AB* be such a numerical power of 3, which is odd by [2], from which the unit *DB* is subtracted. There remains the even number *AD*. If this is divided in half at the point *C*, it follows by [5] that *AC* is equal to the sum of all the terms of the class of powers of 3 preceding *AB*, and this sum is also equal to the number *CD*. It follows by [10] that this sum is divisible by 5, since the number of terms preceding *AB* is 4 or a multiple of 4. It follows that *AD* is divisible by 5, and the same for *CD*. Thus *AD* is not a numerical power of 2, and that is what we wanted to prove.

28. If the unit is subtracted from any numerical power of 3 of any odd rank that does not follow immediately 4 or any multiple of 4, such as the ranks 7, 11, 15, and so on, the only numbers of the class of powers of 3 of which we now need to speak, the remainder is not a numerical power of 2.

Let *AB* be such a numerical power of 3, an odd number by [2], from which the unit *DB* is subtracted, leaving the remainder *AD*, an even number. This number is divided in half at the point *C*, giving the number *AC*, which is equal to the sum of all the terms of the class of powers of 3 preceding *AB* by [5]. From the number *AC* is subtracted the number *EC* equal to the sum of the first two terms, whose sum is 4. There remains the number *AE*, equal to the sum of all the terms preceding *AB*, the first and second excluded. The number of other terms is 4 or a multiple of 4. It follows that the number *AE* is divisible by 8, since by [10] such a sum is divisible by 8. Let the number *FG* be the fourth part of the number *AE*. To *FG* is added the unit *GH*, which is a quarter of the number *EC*. It follows that the number *FH* is odd and is the fourth part of the number *AC* and also the fourth part of the number *CD*. It follows that the number *AD* is divisible by the number *FH*, which is odd, and therefore by [4] the number *AD* is not a power of 2, and that is what we wanted to prove. Therefore we have demonstrated the truth of the negation of the first proposition, and we know that if a unit is subtracted from any numerical power of 3, except those of the third or the second rank, the remainder is not a numerical power of 2.

It remains to demonstrate now the truth of the negation of the second proposition, to show that if a unit is added to any numerical power of 3, with the first term, the unit, and the second term excepted, the sum is not a numerical power of 2.

29. If the unit is added to any numerical power of 3 of odd rank, with the exception of the first, which is the unit, the sum is not a numerical power of 2.

Let *AB* be any numerical power of 3, of odd rank, to which is added the unit *BD*. From the same number is also subtracted the unit *CB*. There remains the even number *AC*, which is divided into two equal parts at the point *E*. This produces the number *AE*, equal to the sum of all the numerical powers of 3 preceding *AB*, by [5]. This sum is even by [5], since the number of terms preceding *AB* is even. To the number *AE*, which is even, is added the unit. The result is the odd number *EB*. But it is known that this result is half of the number *AD*, because the number *AE* is half of the number *AC* and the unit *CB* is half of the number *CD*. It follows that the number *AD* is divisible by an odd number, and therefore by [4], the number *AD* in question is not a numerical power of 2, and that is what we wished to prove.

30. If the unit is added to any numerical power of three of even rank, with the exception of the second rank, the sum is not a numerical power of 2.

Let *AB* be any numerical power of 3 of even rank, which is odd by [2], to which is added the unit *BD*. Let the number *CE* be the last of all the numbers of any rank preceding *AB*. It follows that the number *AB* is triple the number *CE*. From *CE* is subtracted the unit *FE* and let the number *AB* be decomposed as the sum of *AG*, *GH*, *HI*, such that each of these is equal to the number *CF*, and also the remainder *IB*, which is triple the number *FE*, the unit. It follows that *IB* is 3 and therefore *ID* is 4. Let the number *CF*, which is even, be divided into two equal parts at the point *L*. It follows, by [5], that the number *CL* is equal to the sum of all the numerical powers of 3 preceding the number *CE*, and by [7], this sum is even. It follows by [9] that the number *CL* is divisible by 4. It follows that the number *CF* is divisible by 8, because the ratio of *CL* to 4 is the same as the ratio of *CF* to 8. And since the number *AI* is divisible by *CF*, and the number *CF* is divisible by 8, therefore *AI* is divisible by 8. Therefore, by [11], the fourth part of the number *AI* is even. Let this fourth part be denoted by the even number *MN*. To this is added the unit *NO*, which is the fourth part of the number *ID*. It follows that the number *MO* is odd, and is the fourth part of the number *AD*. It follows that the number *AD* is divisible by the odd number *MO*. It follows by [4] that the number *AD* is not a numerical power of 2, and that is what we wanted to prove.

It appears therefore, in terms of this division of numbers into these three classes and their mutual comparison, that, with the exception of the cases noted above, two arbitrary numbers contained in the classes in question are neither equal to each other nor differ by only a unit. Consequently, any two of these numbers, whatever they are, differ by a number, and that is the principal object of our demonstration.

Thus concludes the treatise of Master Levi ben Gershon on the subject of harmonic numbers.

The next three sections contain ideas on amicable numbers (pairs of integers where the sum of divisors of one equals the other, and vice versa). There are a few discussions of amicable numbers in Hebrew medieval arithmetic. These discussions are undoubtedly based on Thābit ibn Qurra's theorem and proof of the primary calculation procedure enabling one to obtain as many pairs of amicable numbers as one wishes [Berggren, 2005, pp. 560–563]. In an elaborate study, [Lévy, 1996] outlines the references to Thābit's results in Hebrew medieval literature and how the Hebrew adaptations of his theorem circulated in Spain, Provence, and Italy in the fourteenth and fifteenth centuries.

5. QALONYMOS BEN QALONYMOS, *SEFER MELAKHIM* (*BOOK OF KINGS*)

This section was prepared by Naomi Aradi

Qalonymos ben Qalonymos ben Me'ir of Arles (1287–ca. 1329), holder of the title *Nasi* and known in Latin as Maestro Calo, was a prolific translator and an original scholar. His surviving original treatises criticize the ethics of his contemporaries, and his translations cover a wide variety of Arab scholarship. He traveled in the Catalan-Provençal area and worked for a time in Rome at the service of Robert d'Anjou, but even though he was a contemporary of Levi ben Gershon, we have no evidence that they ever met.

The earliest Hebrew treatment of amicable numbers is apparently a passage in a treatise that was identified by [Steinschneider, 1870] as the *Book of Kings* by Qalonymos ben Qalonymos. This composition, currently available in two manuscripts, is a compendium consisting of two sections. The first section is an arithmological summary that enumerates universal properties of the first ten numbers and numerical groupings of beings. In the second section specific properties of the numbers are listed, which are reflected in arithmetical statements and algebraic identities in Euclidean style. The discourse on amicable numbers is a part of a cluster of propositions in the second section, which, according to Lévy, corresponds to excerpts in the booklet of Thābit. In this passage the steps of the algorithm for finding pairs of amicable numbers are outlined briefly.

When we want to find amicable[33] numbers, as many as we wish, we set the numbers proceeding from one in a double proportion, including one. The numbers preceding the last number are summed up, including one $[1+2+\ldots+2^{n-1}]$. Then the penultimate number $[2^{n-1}]$ is added to the sum, and the number preceding the penultimate number $[2^{n-2}]$ is subtracted from the sum. The [two] numbers produced by the addition and subtraction [which equal 2^n-1+2^{n-1} and 2^n-1-2^{n-2}] are primes, and neither equals two; if they are not, you proceed until prime numbers come out. Multiply the product of one by the other and by the penultimate number $[\left(2^n-1+2^{n-1}\right)\left(2^n-1-2^{n-2}\right)2^{n-1}]$, and save the result.

Add to the last number $[2^n]$ the fourth number away [in the list of powers of two, namely, 2^{n-3}] (or one, if 1 is the fourth), and multiply the sum by the last number. Subtract [1] from the product, so that the remainder is prime. Multiply this prime by the penultimate number $[2^{n-1}$, yielding $\left(2^n\left(2^n+2^{n-3}\right)-1\right)2^{n-1}]$. The result of multiplication and the saved number each equal the sum of the parts of the other. The numbers produced in this way are called amicable.

[33] *Ne'ehavim*. Literally, "mutually beloved."

6. DON BENVENISTE BEN LAVI, *ENCYCLOPEDIA*

This section was prepared by Naomi Aradi

Another account of Thābit's theorem is found in a text that is said to be a Hebrew translation from the Arabic of the arithmetical part of an encyclopedia attributed to Abū al-Ṣalt (ca. 1068–1134) prepared by Don Benveniste ben Lavi in Saragoza in 1395. Yet [Lévy, 1996] argues that the main part of the Hebrew text (excluding the opening and ending) is in fact a translation from the Arabic of the arithmetical part of Ibn Sīnā's (Avicenna's) encyclopedia *al-Šifā*. The arithmetic section of the treatise covers virtually the same issues as those discussed in the *Arithmetic* of Nicomachus—types of numbers, ratios, proportions, progressions, and the like. The passage on amicable numbers appears as part of the discourse concerning the even-times-even numbers. First, the definition of the relationship between a pair of amicable numbers is illustrated by the example of the numbers 220 and 284. Then a short rendering of Thābit's general algorithm for finding pairs of amicable numbers is given.

The way of generating [amicable numbers] is by adding the even-times-even numbers together with one [$1 + 2 + \cdots + 2^{n-1}$]. If the sum is prime, and on condition that when the last of them [2^{n-1}] is added, or the one before [2^{n-2}] subtracted, the result after the addition and after the subtraction is prime, then multiplying the result after the addition by the result after the subtraction and then multiplying the product by the last number added [2^{n-1}] is a number that is amicable to another [this number being $\left(2^n - 1 + 2^{n-1}\right)\left(2^n - 1 - 2^{n-2}\right)2^{n-1}$].

The number that is amicable to it is the number coming from adding the product of the sum of the above mentioned added and subtracted [numbers $2^n - 1 + 2^{n-1}$ and $2^n - 1 - 2^{n-2}$] by the last number added [2^{n-1}] to the first number that is amicable [$\left(\left(2^n - 1 + 2^{n-1}\right) + \left(2^n - 1 - 2^{n-2}\right)\right)2^{n-1} + \left(2^n - 1 + 2^{n-1}\right)\left(2^n - 1 - 2^{n-2}\right)2^{n-1}$]. These are amicable numbers.

In the margins of the text in [Oxford, Bodleian MS Heb. d. 3/4], a verbose illustration of the general algorithm is given by its application for finding the pair 220 and 284.

A slightly more verbose version of the same procedure appears on another page of the same codex [f. 45r] by the same hand with the following columns of numbers with a derivation of the pair 220 and 284 (compare with Aaron ben Isaac's version below). In anachronistic notation, $a_n = 2^n$, $b_n = 2^{n+1} - 1$, $c_n = b_n + a_n$.

c_n	a_n	b_n
	1	
5	2	3
11	4	7
23	8	15
47	16	13

In the margins of this page, the pair 17,296 and 18,416 is calculated briefly in a different hand as follows: $23 \times 47 = 1081; \times 16 = 17296$ and $16 \times 70 = 1120; +17296 = 18416$. This applies the algorithm above for $n = 5$.

7. AARON BEN ISAAC, *ARITHMETIC*

This section was prepared by Naomi Aradi

Lévy lists a few additional cases in which a very similar short formulation of Thābit's general algorithm is inserted in other Hebrew manuscripts. We complement Lévy's survey with the arithmetic of Aaron ben Isaac (see section I-2)—an elaborate procedure for finding pairs of amicable numbers.

The deliberation on amicable numbers is included in Aaron's discussion of the types of numbers. Unlike both examples above, Aaron describes the stages of the procedure in a practical manner, assisted by a written calculation table. The practical instructions are demonstrated by detailed examples in which two pairs of amicable numbers are calculated (220 and 284; 17,296 and 18,416). However, using his own guidelines Aaron found another pair of numbers (2024 and 2296) that are not amicable, a fact that he apparently failed to notice. Although Aaron must have drawn the basic algorithm from external sources, perhaps this error could testify to his attempt to implement the algorithm by self-instruction.

[The amicable numbers] are any two numbers, such that the [sum of the] whole parts of each equal the other. You find the first [of each pair] in one manner, and the partners in another. The foundation of both is the doubled numbers called even-times-even, from which we produce primes, and from the primes the amicable numbers.

We first order the doubled numbers, which are even-times-even, starting from one as far as you wish. Then take [the sum of the first two doubles] 1 and 2, which are 3, [and set it] against 2. Take [the sum of the] three doubles from 1 to 4, which are 7, [and set it] against the last double you took, which is 4. Take [the sum of the] 4 doubles from 1 to 8, which are 15, and set it against the last double, which is 8. Take [the sum of the] 5 doubles from 1 to 16, which are 31, [and set it] against 16. [Set] also 63 against 32, 127 against 64, [and 255 against 128.]

A corrupt passage follows, indicating that the third column is the difference between the number in the second column and the number above the corresponding number in the first column. The first column in the diagram below is a reconstruction. In anachronistic terms, $a_n = 2^n$, $b_n = 2^{n+1} - 1$, $c_n = b_n - a_{n-1}$.

a_n	b_n	c_n
2	3	2
4	7	5
8	15	11
16	31	23
32	63	47
64	127	95
128	255	191

From the primes in the third column [c_n] we get the first of each pair of amicable numbers.

The text also notes that 95 will not produce amicable numbers, because it is not prime. The following passage is not quite clear. It instructs the reader to take pairs of successive primes from the third column, but it does not indicate that they are to be multiplied by each other and by the corresponding number in the first column. The examples below, however, fill the gap. The formula is $c_{n+1} \times c_n \times a_n$.

The second [of each pair of amicable numbers] is produced differently. Take the double which follows the double from which you produced the first amicable number [a_3], and add one, which is the first in actu of even-times-evens [a_0]; to the second doubled number of the seconds [a_4] add 2 [a_1]; to the third [a_5], add 4 [a_2], and so on. Then multiply [a_{n+1} by $a_{n+1} + a_{n-2}$], and subtract one from the result. You get a prime. Multiply it by the doubled number preceding it in nature [a_n], and the result is the partner [yielding $(a_{n+1}(a_{n+1}+a_{n-2})-1)\,a_n$].

I hereby give six examples, three for the first amicable numbers, and three for their partners.

In a corrupt passage, Aaron indicates that the partner of an amicable number can also be derived by summing the parts of the amicable number.

First example: If we want to find the first amicable number of the first pair, we take what remains from 7 and 15 as you see in the third column [of the table], which are the prime numbers 5 and 11 [c_2 and c_3, respectively]. We multiply them by each other, yielding 55. We then multiply 55 by the double number 4 [a_2], yielding 220. This is the first amicable number of the first pair.

By summing the parts of 220, Aaron finds its partner, 284. The next examples for the first numbers of amicable pairs according to this method are $c_4 \times c_3 \times a_3 = 2024$ and $c_5 \times c_4 \times a_4 = 17296$. Aaron fails to note that the first of these does not work, because the construction of the partner involves a non-prime.

We wish to find the partner of the first amicable number we have. Here is how. Take the double number 8 [a_3], and multiply it by itself with 1 added, which is 9, yielding 72. Subtract 1 from 72, there remain 71, which is prime. Multiply it by the doubled number naturally preceding it, which is 4 [a_2], yielding 284. This is the partner of the first amicable number.

Aaron lists the parts of 284 and then produces the next two partners: $(a_4(a_4+a_1)-1)\,a_3 = 2296$ (which fails because the first factor, 287, is not prime) and $(a_5(a_5+a_2)-1)\,a_4 = 18416$.

III. MEASUREMENT AND PRACTICAL GEOMETRY

This section opens with two important treatises that summarize geometric knowledge in a semi-practical style. This does not mean that the authors or readers were ignorant of higher geometry, but that the treatises focused on unmotivated rules or on forms of reasoning somewhat more intuitive than the highbrow Euclidean style. *The Book of Measure* does not provide reasoning at all, but the problems solved are clearly not all practical (one is not likely to know the area of a rectangle and the sum of its side and diagonal without knowing its sides in a practical context). Bar Ḥiyya's *The Treatise on Measuring Areas and Volumes* does provide solid proofs, but it simplifies the Euclidean presentation considerably.

We continue with two discussions of measurement in religious context, which show the relevance of practical geometric reasoning for religious scholars. To conclude, we quote from Levi ben Gershon's discussion of iterative linear interpolation and trigonometry as presented for application in astronomy.

1. ABRAHAM IBN EZRA (?), *SEFER HAMIDOT* (*THE BOOK OF MEASURE*)

This treatise came down to us in Hebrew and in Latin translation. The only surviving Hebrew manuscript attributes it to Ibn Ezra (see section I-1), but this attribution is not considered certain. The treatise opens with an unsystematic collection of notes on arithmetic, which, according to a conjecture in [Lévy and Burnett, 2006], may have been an early draft or collection of notes for Ibn Ezra's *The Book of Number*. It then goes on to treat geometry. The treatise devotes a chapter to measuring triangles, then one to quadrilaterals, another chapter to circles and trigonometry, another to measuring solids, and a final chapter to the use of the astrolabe for determining distances and heights.

The book provides hardly any proofs or arguments, but it summarizes techniques for obtaining geometrical measurements from geometrical givens. It contains no algebraic language and no arsenal of Euclidean building blocks to work with, but the procedures it includes seem to depend on both. [Lévy and Burnett, 2006] conjecture that the book was to be revised and improved, but was then abandoned when Ibn Ezra became acquainted with Bar Ḥiyya's work on geometry (see below).

Preliminary definitions (2.1–2.4)

Because all measurements depend on number, I shall indicate the principles.

The point is in the mind, not in the [drawn] figure. Between two points there is a line; this is "length." Between two lines there is breadth; this is "surface." Likewise, when [one considers] the vertical [dimension], there arises depth, and thus "body" [a solid].

Know that among measurements, there is sometimes a line, at other times a surface, at other times a body.

I shall start now with the measurements of surfaces. Although the [basic] measurement of the area is the square, I shall start by mentioning the triangular figures, because the triangular figure is the principle of all [rectilinear] figures and every figure returns to it.

Triangle measurement (3.1–3.26)

The triangular figure of which the [three] angles are acute and the [three] sides equal [i.e., the equilateral triangle].

Subtract from the square of the side its quarter; its root is the height. The area: multiply the height by half the side, or the side, taken as the base, by half the height. Or take of the square of the height five of its ninths and a fifth of its ninth. Or take of the square of the side its third and add to it its tenth.

Or add the three sides and take the half; observe by how much this half [of the perimeter] exceeds each side, and multiply the excess by itself—this [means] taking its square—and multiply by the first line, which is the excess of the half [of the perimeter], and then one gets a cube; multiply this cube by half the sides [the half-perimeter]; take the root of the product, and in this way [you have] the area.[1]

Problem: The area [of an equilateral triangle] is so much. How much is the side? Take the square of the area; multiply this square by three, take the root of the product and add to it its third; take the root of this sum, and this is the side.

Another: The area is so much. How much is the height? Take a third of the square of the area; take its root and the root of the root, and divide the area by it; and you will find the height.

[1] This is Heron's formula applied to an equilateral triangle.

. . .

The triangular figure of which the angles are acute, the base different [from the two sides], and the two sides equal [i.e., an isosceles triangle]. From the square of one side, subtract the square of half the base; the root of the remainder is the height. The area: multiply the height by half the base, or vice versa.

Problem: The area [of the isosceles triangle] is so much, the height is so much. How much is the base? Divide the area by half the height.

. . .

Another: The base is so much, the height is so much. How much is the side? Add the square of the height to the square of half the base, and take the root of the sum.

. . .

Another: The area is so much, the base exceeds the height by so much. How much is the base and how much is the height? Double the area and take the square of half the excess; add them and take the root of the sum; add to the result half the excess; you will find the base. In the same way, subtract half the excess from the root; you will then find the height.

The triangular figure of which the angles are acute and the sides are different [in length, i.e., a scalene triangle].

Make the base whichever side you wish; take the squares of the other two sides; subtract the smaller from the larger; divide half the remainder by the base; add the result [i.e., the quotient] to half the base. You will then find the larger segment of the fall.[2] If you subtract half the result from half the base, you will then find the smaller segment of the fall. Take the square of the larger segment of the fall; subtract from the square of the longer side [i.e., the longer of the two sides that are not the base]; the root of the remainder is the height [in respect to the base]. Or subtract the root of the smaller segment of the fall from the square of the shorter side; take the root of the number [obtained]; this is [also] the height. It is always the same. The area: by multiplying half the height by the base, or vice versa.

The triangular figure with one right angle and two acute angles.

The area of this figure: multiply half of one short side by the other short side, the two short [sides] being opposite the long side, which is the diagonal.

Problem: [In a right-angled triangle] one of the two sides is so much, the other so much, these two being the short [sides]. How much is the long side? Add the squares of the two [short sides]; the root is the long side.

Another: The area is so much, one of the short [sides] exceeds the other by so much. How much is each side? Double the area; take the square of half the excess; take the root of their sum; add half the excess to the root; you will have one of the [short] sides; and if

[2] The "segments of the fall" (*mekhona*) are the parts of the base as divided by the perpendicular from the opposite vertex.

you subtract from the root [half the excess], you will have the other [short side]. Add the square of this to the square of that; the root of the sum will be the long side [hypotenuse].

. . .

Square and rectangle measurements (4.1–4.25)
The quadrilateral figure of which the length is the same as the breadth and all the angles are right angles [= the square].

Take the square of one side. This is the area.

Problem: We have multiplied the area by so much; the result divided by one side gives so much; what is the length of each side? Divide the result by the number by which the area had been multiplied.

Another: The diagonal is so much. How much is the side? Take the root of half the square of the diagonal.

. . .

Another: We have added the sides and the area; this gives so much. How much is the side? Take the square of half the number of all the sides [= 4] and add it to the sum [of the area plus the four sides]; subtract from the root of this result half the number of the sides [= 2].

Another: We have subtracted the sides from the area. The remainder is so much. How much is the side? Add the square of half [the number] of the sides to the remainder; take its root and add to it half the number of the sides.

. . .

The quadrilateral figure of which the length is greater than the width and the angles are right angles [=the rectangle].

[To obtain] the area, multiply the side of the length by the side of the breadth.

. . .

Another: The area is so much, and the longer side exceeds the shorter by so much. How much is each of the sides? Take half the excess [of the longer side over the shorter]; then take its square; then add this to the area and take the root of the result and add to it half the excess. The longer side is so much. If, in the same way, you subtract [half the excess] you will find the shorter side.

Another: We have added the longer side to the diagonal; this comes to so much; the shorter [side] comes to so much. How much is the longer side and how much the diagonal? Take the square of the sum [of the diagonal and length]; subtract from it the square of the known side; take half the remainder, and divide it by the initial sum; you will find the longer side. If you subtract it [the longer side] from the sum mentioned, you will find the diagonal.

Or take the square of the shorter side, divide it by the initial sum [of the diagonal and length] and add to the result [the quotient] the initial sum; half [this new sum] is the diagonal, from which you will know the [longer] side.

Another: We have added the two sides; they come to so much; the diagonal is so much. How much is each of the sides? Subtract from the square of the sum [of the sides] the square of the diagonal; half of the remainder is the area.[3]

. . .

Another: We have added the two sides and they exceed the diagonal by 4, and the area is 48. How much is the diagonal and how much each side? Take the square of 4 and subtract it from twice the area; divide half the remainder by 4 and you will find the diagonal. One will proceed in the same manner when the question is: "The area is so much, the diagonal is so much. How much is each side?"

Another similar [problem]. We have subtracted [all] the sides from the area and the remainder is 20, and one of the sides exceeds the other by 2. How much is each side? Double the 2, add it to the 20; take the number of sides—which is 4—and subtract from this the excess [of the longer side over the shorter]; the remainder is 2; take half of it, then [take] its square, which is one; add it to 24, which makes 25, whose root is 5. Add to it half of 2; so much is the shorter side [= 6].[4]

. . .

Another: We have added [all] the sides and the area, and they come to 76, and one of the sides exceeds the other by 2. How much is each side? Double the 2 and subtract it from the number 76; one knows that the number of the sides is 4; add to them 2, which is the excess of the length over the breadth; this makes 6; take the square of half this amount; add it to 72, and take its root [= 9]; subtract from it 3, which is half 6; thus one has the shorter side.[5]

. . .

Parallelogram measurements (4.44–4.47)

The quadrilateral figure, similar to that which was mentioned above, of which the [opposite] angles are equal, but of which two sides are equal and the other two [equal but] different [from the first two, i.e., a parallelogram that is not a rhombus].

[3] The text does not contain the completion of the solution, although if one knows the sum of the sides and the product, there is a standard method for finding the two sides.

[4] Although the answer is correct, the method is not at all clear. But the author appears to be reducing the problem to one already solved, namely, determining the sides when the area and the difference of the sides are known. In fact, the instruction to "double the 2 and add it to the 20" converts the problem to the system we would write as $x(y-4) = 24$ and $x = (y-4) + 2$, for which the standard algorithm presented above applies to determine x.

[5] As in the previous problem, the instructions seem to convert the problem to the system $x(y+4) = 72$, $(y+4) = x+6$, to which a standard algorithm applies.

Problem: The longer side is 9, the shorter 5; the longer diagonal is the root of 160 and the shorter one, the root of 52. [What is the area?] Take the square of each of the two sides that differ from each other and add them; this makes 106. Subtract this from the square of the longer diagonal; the difference is 54. Take half of this, which is 27. Divide this by the longer side, which gives 3. Take the square of this, which is 9, and take the square of the shorter side, i.e., 25. Subtract the smaller from the larger. There remains 16, of which the root is 4. If one multiplies this by 9, which is the longer side, it comes to 36. This is the area.[6]

Or take the squares of the two sides and add them. They make 106. Subtract them from the square of the longer diagonal; the remainder is 54. Take half of this, i.e., 27; divide this by 5, i.e., the shorter side [$= 5\frac{2}{5}$]; take the square of the result [$= 29\frac{4}{25}$] and also the square of 9, which we already have [the longer side], and subtract the smaller from the larger [$= 51\frac{21}{25}$]. Take the root of the remainder, i.e., $7\frac{1}{5}$, and multiply this by 5, which is the shorter side. This makes 36, which is, again, the area.

. . .

Circles and circular arcs (5.1–5.27)

The circle.

The area of the circle. You subtract from the square of the diameter its seventh and half of its seventh. Or multiply the square of the diameter by 11 and divide by 14. Or multiply half the diameter by half the circumference. Or a quarter of the diameter by the whole circumference. Or the diameter by a quarter of the circumference.

If you know the diameter, multiply by $3\frac{1}{7}$. You will always find the circumference. Or multiply the diameter by 22 and divide by 7.

If you know the circumference, multiply it by 7 and divide by 22. You will have the diameter.

. . .

Problem: We have subtracted the diameter from the circumference, the remainder is so much. How much is the circumference? And how much is the diameter? Divide the remainder by $2\frac{1}{7}$ and you will find the diameter. Or multiply by 22 and divide by 15 and you will find the circumference. . . .

The arc [or the circular segment].

If one is dealing with a semicircle, its area is like that of half a circle. If it is smaller or larger [than a semicircle], you must know the diameter of the circle from which the circular segment has been cut, and the length of the chord of the arc and of the sagitta.[7] When you know two of these [three] elements, you can determine the third.

[6]To find the area of the parallelogram, one has to find the height. The procedure here and below is similar to that of finding the height and “segment of the fall” in a scalene triangle, described above, except that in this case, the triangle is obtuse.

[7]The sagitta is the segment of the diameter perpendicular to the chord lying inside the circular segment.

Problem: The chord is 8, the diameter, 10. How much is the sagitta? Subtract from the square of half the diameter the square of half the chord; take the root of the remainder, and subtract it from half the diameter; you will find the sagitta [= 2].[8]. . .

Another: The sagitta is 2, the chord, 8. How much is the diameter? Take the square of half the chord; divide it by the sagitta, and add the result to the sagitta; you have the diameter [=10].[9]

. . .

[Table of Sines]

The arc is 90 [degrees] and the Sine 60.[10] If you take the root of half the square of 60, you will find the Sine of the arc of 45 [degrees]. If you subtract the square of half the Sine [of 90 degrees] from its [whole] square, and take the root of the remainder, you will have the Sine of the arc of 60.

The Sine of one degree amounts to one unit and three minutes [1;3].
The Sine of an arc of 5 [degrees] is 5 and 14 minutes [5;14].
The Sine of an arc of 10 is also 10 and 25 minutes [10;25].
The Sine of an arc of 15 is 15 and 32 minutes [15;32].
The Sine of an arc of 20 is also 20 and 31 minutes [20;31].
The Sine of an arc of 25 is also 25 and 21 minutes [25;21].
The Sine of an arc of 30 is also 30 with no subdivisions [30;0].
The Sine of an arc of 35 is 34 and 25 minutes [34;25].
The Sine of an arc of 40 is 38 and 34 minutes [38;34].
The Sine of an arc of 45 is 42 and 25 minutes and 35 seconds and 4 thirds [42;25,35,4].

If you double it and turn the degrees into minutes, the minutes into seconds, the seconds into thirds and the thirds into quarters, you will then find the [square] root of 2 with great precision.[11]

The Sine of an arc of 50 is also 45 and 58 minutes [45;58].[12]
The Sine of an arc of 60 is also 51 degrees and 58 minutes [51;58].
The Sine of an arc of 65 is 54 and 23 minutes [54;23].
The Sine of an arc of 70 is 56 and 32 minutes [56;32].

[8] If d is the diameter, s the sagitta, and c the chord, the Pythagorean Theorem gives $\left(\frac{d}{2}\right)^2 = \left(\frac{d}{2} - s\right)^2 + \left(\frac{c}{2}\right)^2$, which is the identity used here.

[9] *Elements* III. 35 states that the product of the two segments of the diameter determined by the chord is equal to the square on half the chord. Thus, if d is the diameter, s the sagitta, and c the chord, we have $s\,(d - s) = \left(\frac{c}{2}\right)^2$.

[10] The Sine of an arc in medieval literature is half the chord of double the angle in a circle of given radius. Thus the Sine of a 90° arc is equal to the radius, which implies that for this table, the radius is 60; this value is often called the “whole Sine.” The two calculations in this paragraph are straightforward.

[11] The Sine of a 45° arc is $60\frac{\sqrt{2}}{2}$. Doubling this and changing degrees to minutes, etc., will therefore give $\sqrt{2}$ in sexagesimal form $1; 24, 51, 10, 8 = 1.41421356\ldots$ in decimal form.

[12] The value for 55 is missing.

The Sine of an arc of 75 is 55[13] and 58 minutes [55;58].
The Sine of an arc of 80 is 59 and 8 minutes [59;8].
The Sine of an arc of 85 is also 59 and 46 minutes [59;46].

If you take [an arc of] more than 60 [degrees], consider what is needed to complete an arc of 90 [degrees]. Subtract the square of its Sine from the square of the whole Sine, and take the root of the remainder. You will find what has [already] been indicated.

If you want to know the Sine of an arc that does not appear [in the table], you work it out proportionally: look at what represents [the difference of] 5 [degrees] of arc for the Sine, and consider the proportion that you have to add to the number you already have.[14]

Pyramid measurement (6.11–6.13)

A solid of which the top is square and different from the base, whose breadth is equal to its length [a truncated right pyramid or frustum].

[The side of the square] of the top is 2, that of the base, 4, the height, 10, and it is the height. Subtract the top from the base; observe what is the ratio between the remainder and the top—they are equal [2]; one can, then, complete this figure to make it 10, which is like the [original] height [to make a complete pyramid with the height 20]. The volume [of the pyramid thus obtained] in completing the height [of the truncated pyramid] will be $106\frac{2}{3}$ [$= \frac{1}{3} \times 16 \times 20$]. Let us then make the top the base [of a small pyramid]. The volume of this complement will be $13\frac{1}{3}$ [$= \frac{1}{3} \times 4 \times 10$], which one subtracts from that which has been given above. The remainder is $93\frac{1}{3}$. Or let us add the square of the top to the square of the base; let us multiply the side of the base by the side of the top; let us add this result to the squares indicated; and let us multiply the sum of them all by a third of the height; one finds the volume.[15]

Measurements of heights and distances using an astrolabe quadrant (7.1–7.13)

To measure the height of hill, a tree or a tower.[16]

Set the alidade [lit: "line"] of the astrolabe [lit: "instrument of brass"] [in the quadrant] on which you have graduated the degrees of the sun[17] at 45 degrees [Fig. III-1-1]. Go forwards or backwards until you can see the top [of the object to be measured] in the alignment of the two holes of the alidade. Then measure the distance which separates your feet from the foot of the tree, the hill or the tower. Add to it the height between your eyes and the ground. You obtain the measurement that you were looking for [Fig. III-1-2]. You will measure it in cubits, palms, or any other unit of measure.

Or when you have seen the top [of the object to be measured], turn and, without moving from the place where you are, set the alidade on 45 degrees in the lower quadrant, . . .

[13] This should read 57.

[14] The procedure suggested here seems to be linear interpolation.

[15] This procedure is in the *Moscow Mathematical Papyrus* (see [Imhausen, 2007, p. 33]).

[16] The descriptions that follow are the standard ways used in the Medieval period to measure unknown heights and distances. See sections I-4-5 and II-5-5 in Chapter 1 for other examples.

[17] Since the astrolabe is typically used to measure the altitude of heavenly bodies, including the sun, in degrees, the author is calling the markings on the circle "the degrees of the sun." In this case, he is just instructing us to set the alidade at 45° so that we obtain an isosceles right triangle.

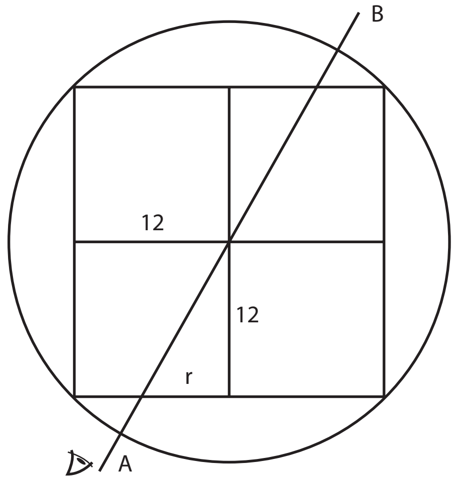


Fig. III-1-1. Diagram of astrolabe. *AB* represents the alidade, and r, measured on a scale of 12, is the gnomonic umbra recta.

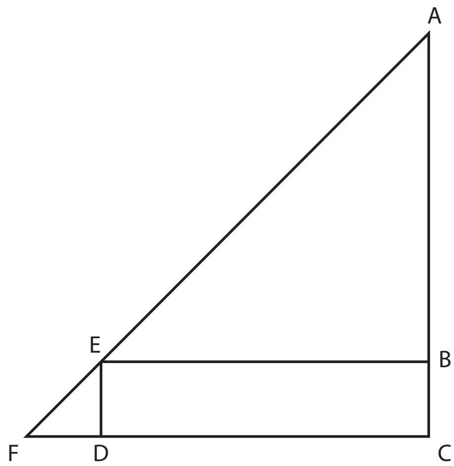


Fig. III-1-2. *A* is the top of the hill, *E* is the eye of the observer, and angle $AEB = 45^\circ$. Thus the height of the hill is $AC = AB + BC = CD + ED$. Alternatively, the astrolabe can be turned around to locate *F* and obtain *AC* as $CD + DF$.

and observe through the hole the point on the ground; then measure the distance that separates that point from the foot of the tree or the tower; you will thus find what you are looking for [Fig. III-1-2].

Or suspend the instrument from your right hand and observe how the top [of the object to be measured] looks; then read off [on the astrolabe] the gnomonic umbra recta; determine the ratio of 12 to the "number of the umbra" [r]; observe the same ratio between [the distance between the top and the horizontal line issuing from your eyes, h, and] the

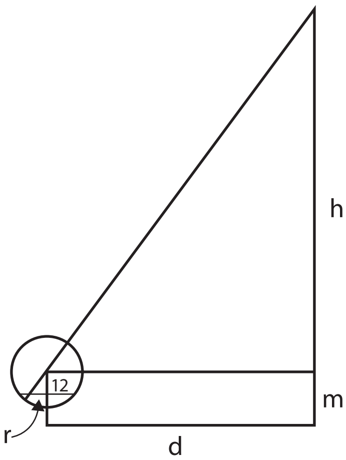


Fig. III-1-3. 12:$r = h$:d, so $h = 12d/r$ and the height of the hill is $h + m$.

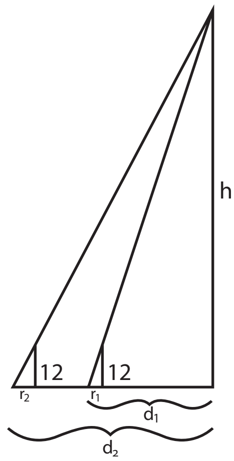


Fig. III-1-4. $h = 12d_1/r_1 = 12d_2/r_2$. If $\Delta = d_2 - d_1$, then $h = 12\Delta/(r_2 - r_1)$, as stated. Also, $di = ri\Delta/(r_2 - r_1)$ for $i = 1, 2$.

distance that separates your feet from the place that is sought [d]; add [to the distance obtained by this proportion] the height [m] between your eyes and the ground [Fig. III-1-3].

If you cannot reach the foot of the tower, suspend the instrument from your right hand and read off the degrees of the height [of the tower] from any place, and know the gnomonic umbra; then, go forwards or backwards and for a second time take the degrees of the top, and know the gnomonic umbra, and know how many graduations separate the two umbras [measured on the astrolabe]. Then multiply by 12 the cubits or palms that you have passed through between the two positions, and divide the product by the difference between the two umbras; add to this that of the height between your eyes and the ground; such is the measurement that you are looking for [Fig. III-1-4].

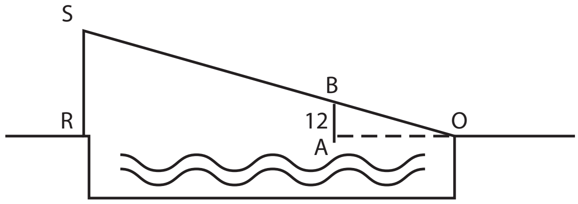


Fig. III-1-5. *OA*:*AB* is the ratio of the gnomonic umbra to 12. This is the same as the ratio *OR*:*SR*, where *SR* is the height of your body.

If you want to know at what distance you were from the base of the object to be measured in the first position, multiply the cubits that you have passed through in going forwards or backwards by the first umbra, and divide the product by the difference between the two umbras. The product is the distance [of the foot of the tower] to where you were in the first position. Similarly, multiply the cubits by the second umbra and divide the product by the difference; you will find the distance relative to the second position.

. . .

To measure the breadth of a river, stand on the bank and suspend the instrument from your left hand; raise and lower [the astrolabe] so that you can see the other bank of the river; then read off the gnomonic umbra; compare the ratio of the measurement to 12. Having obtained the result, multiply this ratio by the height of your body; such is the measure [of the breadth] of the river [Fig. III-1-5].

. . .

2. ABRAHAM BAR ḤIYYA, *ḤIBUR HAMESHIḤA VEHATISHBORET* (*THE TREATISE ON MEASURING AREAS AND VOLUMES*)

Abraham bar Ḥiyya (ca. 1065–1145) was based in Barcelona, where he worked as a scholar and community leader.[18] His Jewish title was *nasi* (honorary leader), and Arabic title *ṣāḥib ash-shurṭa* (head of the guard, transliterated in Latin as “Savasorda”). He wrote on mathematics, astronomy, astrology, and philosophy, and he is recognized as being the first Jewish scholar in the Arabic-speaking world to write on science in Hebrew. This choice of language was, at least in part, due to the Jewish community’s lack of access to the Arab language in Provence, where he visited. His work includes translations from Arabic to Hebrew, and he collaborated with Plato of Tivoli on translations into Latin as well.

Bar Ḥiyya’s mathematical work includes *The Foundations of Wisdom and Tower of Faith*, an encyclopedia of which only the introduction and mathematical parts survive,[19] and *The Treatise on Measuring Areas and Volumes*, from which the following selection is taken.

[18] For general information, see [Steinschneider, 1893–1901; Lévy, 2001, 2008; Langermann, 2007].

[19] A Hebrew-Catalan edition was published in [Millás Vallicrosa, 1952]; subsequent research is available in [Levey 1952, 1954; Rubio, 2000].

This book was translated into Latin in 1145 by Plato of Tivoli under the title *Liber Embadorum*, and made an impact on European scholarship.[20]

The Treatise on Measuring Areas and Volumes opens with a motivational introduction, pointing out the value of geometry for secular and holy affairs. The first book then gives some basic definitions and theorems that serve as building blocks for the rest of the book. These are mostly theorems from Euclid's *Elements*, Books II and III, along with a basic discussion of similarity and congruence. The treatment is usually more intuitive than Euclid's and is accompanied by arithmetic examples.

Book II is the core of the work. Its first part addresses measurements of squares, rectangles, and rhomboids (deriving their areas, sides, diagonals, etc. from one another), and includes a geometric treatment of quadratic problems. The second part deals with triangles, and the third with general quadrilaterals. Part four deals with circle measurements and includes a trigonometric table for calculating arcs from chords. The fifth part studies the measurement of polygons by triangulation; it ends with some practical notes on measuring sloping and hilly lands.

Book III is a simplified version of Euclid's book on equal division of plane areas. It is one of the few witnesses to this lost work.[21] Book IV provides a brief treatment of solids. The final section deals with practical tips for land measurement, including the use of an instrument shaped as a right angle, and concludes by repeating the warning against simple but false rules of thumb.

This work is not a fully scholarly geometry exposition but a compromise between an introduction to abstract geometry and a measurement manual. It provides a good intuitive introduction to geometrical reasoning and some elementary tips for land measurements (excluding the use of protractors or astrolabes). As the selection below shows, it diverges in several ways from the scholarly Arabic tradition and is probably closer to the popular *muʿāmalāt* tradition, whose Western branch does not survive in Arabic manuscripts.

Motivations for studying geometry (Introduction)

This section discusses the secular and holy motivations for studying geometry, and it warns against rule-of-thumb approximations that may result in unjust distributions of property.

[The scriptures say] "I the Lord am your God, instructing you for your own benefit, guiding you in the way you should go,"[22] that is, instructing you in whatever is useful for you, and guiding you on the way you follow, the way of the Torah. From which you learn that any craft and branch of wisdom that benefit man in worldly and holy matters are worthy of being studied and practiced.

I have seen that arithmetic and geometry are such branches of wisdom, and are useful for many tasks involved in the laws and commandments of the Torah. We found many scriptures that require them, such as "In buying from your neighbor, you shall deduct only for the number of years since the jubilee," and "the more such years, the higher

20 See [Curtze, 1902] for an edition of the Latin version with German translation as well as [Millás Vallicrosa, 1931] for an edition in Catalan.

21 See [Archibald, 1915] for a reconstruction of this treatise and [Hogendijk, 1993] for an edition of the Arabic version.

22 Isaiah 48:17.

the price you pay; the fewer such years, the lower the price," followed by: "Do not wrong one another, but fear your God."[23] But no man can calculate precisely without falsification unless he learn arithmetic. . . . Moreover, the Torah requires geometry in measuring and dividing land, in Sabbath enclosures and other commandments.[24] . . . But he who has no knowledge and practice in geometry cannot measure and divide land truly and justly without falsification. . . . It suffices to note that the blessed God prides himself in this wisdom, as is written: "He stood, and measured the earth"[25] and "Who measured the waters with the hollow of His hand, and gauged the skies with a span."[26] So you see from these writings that the blessed God created his world in well founded and weighed out measurement and proportion. And a man must be like his creator with all his might to win praise, as all scholars agree, so from all this you see the dignity of these branches of wisdom. He who practices them does not practice something vain, but something useful for worldly and holy matters.

As I see it, Arithmetic, which is useful for worldly matters and crafts as well as for the practice of many commandments, is not difficult to understand, and most people understand it somewhat and practice it, so one does not need to write about it in the holy tongue. Geometry is also as useful for as many matters as arithmetic in worldly matters and commandments from the Torah, but is difficult to understand, and is puzzling to most people, so one has to study and interpret it for land measurement and division between heirs and partners, so much so that no one can measure and divide land rightfully and truthfully unless they depend on this wisdom.

I have seen that most contemporary scholars in *Ṣarfat*[27] are not skillful in measuring land and do not divide it cleverly. They severely belittle these matters, and divide land between heirs and partners by estimate and exaggeration, and are thus guilty of sin. . . . Their calculation might mete out a quarter to the owner of a third, and a third to the owner of a quarter, and there is no greater theft and falsification.

If one says that our fathers cared little for calculations and precision, as they say: "If the side of a square is a cubit, its diagonal is one and two fifths,"[28] whereas in such a square, with equal sides and right angles, for each cubit in the sides the diagonal, calculated precisely, is a cubit and two fifths of a cubit and one in seventy parts of a cubit and a small excess [Bar Ḥiyya goes on to quote further Talmudic examples stating that the ratio between the circumference and diameter of a circle is 3, that the excess of the area of a square circumscribing a circle is 1/4, and that the excess of the area of a circle circumscribing a square is 1/3]; so one of our contemporaries, who belittles the division and measurement of land, may wrongfully exclaim: "from the words of our fathers we learn that calculations need not be too precise; we should learn from them, follow them,

[23] Leviticus 25:15–17.

[24] The text here quotes Numbers 35:5, Deuteronomy 21:2, and Numbers 26:53–54.

[25] Habakkuk 3:6. Here, exceptionally, we use the Bible of King James, as the Jewish Publication Society version interprets the root *mdd* differently, translating the verse as "When He stands, He makes the earth shake," which is not Bar Ḥiyya's reading.

[26] Isaiah 40:12.

[27] The term *Ṣarfat* used here referred at the time to various areas of the Catalan-Occitan region.

[28] Talmud, *Suka* 8a.

and calculate by estimate and approximation without study and precision. So you cannot protest and refute us."

I will reply to him and say: "God forbid! Our fathers did not allow us to dismiss calculations, nor steal from heirs, nor give any of them more or less than their fair share, as you sinfully do today. Even though they calculated the diagonal of the square and the diameter and circumference of the circle and the area between the circle and the square imprecisely, as you say, nevertheless they warned us and gave us strict orders against stealing and falsifying in measuring land.[29] . . .

All the calculations that you mentioned, where our fathers were imprecise, are completely harmless. Indeed, the approximation of the diagonal of the square is harmless in measuring and dividing land, because you don't need the diagonal when measuring a square. And the calculation of the excess of a square over a circle and vice versa and the diameter and circumference of a circle are required only for Sabbath enclosures and for keeping different kinds of seeds and crops apart. Now anything involving this [Talmudic] approximation renders the commandments obeyed more strictly, rather than more leniently, and does not harm anybody's property. Moreover, in most places where our fathers approximate imprecisely, they make a note, such as "he only gave an approximate figure; and in this case it is in the direction of stringency."[30] . . . This is unlike contemporary errors, which cause harm and great losses to people's property.

Geometric building blocks (from Book I)

The following is a version of Euclid's *Elements* II.4. But note that the diagram is different from that of Euclid, that the proof relies on elementary cut-and-paste rather than on Euclid's bisection of a parallelogram by its diagonal, and that the entire discussion is accompanied by an arithmetical interpretation. This approach represents a line of transmission or pedagogical tradition different from the scholarly Euclidean one, and it characterizes most theorems in the first book of the treatise. The results in this book serve as "building blocks" for the solution of problems in the subsequent books.

I say that in any line divided into two parts, the square[31] of the line by itself equals the square of each part by itself and double the rectangle of one of its parts by the other. I give you an arithmetical example. Let this line be 12 cubits in magnitude, and divide it into two parts, 7 and 5. The square of the whole line by itself is 144. This number equals the square of 7 by itself, which is 49, and the square of 5 by itself, which is 25 (together they make 74), and double the rectangle of 7 by 5, which is 70.

To give an example with a diagram, set the line *AB* of magnitude 12, and divide it at *E* [Fig. III-2-1]. The large part, *AE*, is 7 cubits, and the small part, *EB*, is 5 cubits. We construct the square of the line by itself, making *ABCD*, a square quadrilateral with equal sides and right angles (as all its sides are of length 12) and with area 144 cubit by cubit.

[29] The text here quotes Leviticus 19:15 and the Talmud, *Baba Meṣiʿa* 107b and 61b.

[30] The text quotes here the Talmud, *Suka* 8a.

[31] The Hebrew uses two terms: *merubaʿ*, which is a general term for a quadrilateral, and *ribuaʿ*, which is the noun for the action of making a *merubaʿ*. Both terms are used here in geometric as well as arithmetic contexts (numerical products and squares). A strict translation would use "quadrilateral" or "the quadrilateral of" throughout. For the benefit of the reader, however, I used "quadrilateral," "rectangle," and "square," according to the context. But note that this choice subtracts from the arithmetic overtones of the book.

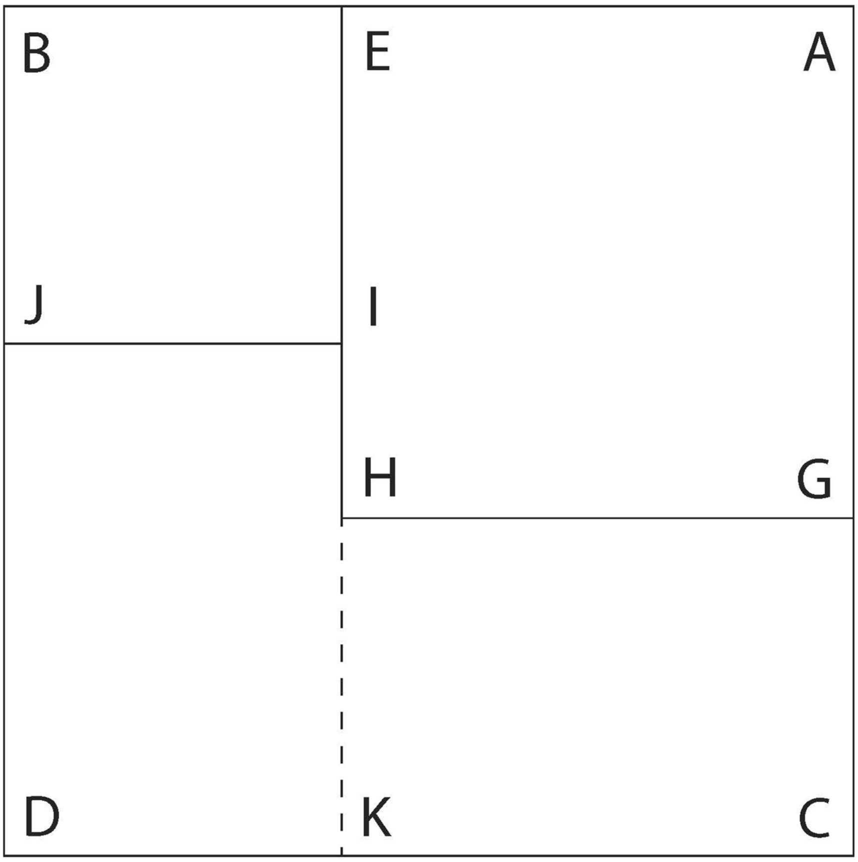


Fig. III-2-1.

We construct the square of the large part by itself, which is the rectangle *AEGH*, and the small by itself, which is *EBIJ*. To complete the large square there remain the rectangle *GHCK* and the rectangle *IJDK*. And we know that both these rectangles are the rectangles of one part by the other, which is 5 times 7. Indeed, the line *AGC* equals the line *AEB*, and the line *AG* equals the line *AE*, which is the large part, leaving the line *GC* that equals the line *EB*. Moreover, the line *CKD* equals the line *AEB*, and the line *CK* equals the line *AE*. So we find that the rectangle *CGHK* is the rectangle of the large part by the small. Further, the line *KD* equals the line *EB*, and the line *JD* equals the line *AE*, which is also the large part by the small. We find that the square of the large part, which is *AEGH*, and the square of the small part, which is *EBIJ*, and twice the rectangle of the large part by the small, which are the rectangles *GHCK* and *IJKD*, complete the large rectangle *ABCD*, as you see in this diagram.

Geometric presentation of quadratic equations (from Book II)

Interpreted algebraically, this section solves the standard compound quadratic equations and justifies the solutions geometrically. One should, however, note its divergence from the Arabic algebraic tradition. First, these problems are not set here as standard problems to which other problems should be reduced. They appear without any distinction among a list of other geometric problems. The choice of 4 (the number of sides) as the coefficient of the linear term might make them appear less generalizable than they actually are. Moreover, the language is entirely geometric, without any special word to designate the unknown side. Finally, the Arabic tradition follows a rather strict normalization of the canonical quadratic problems, where only additive terms are equated; here two problems equate the difference between the area and the sides with a given number. As above, this suggests a line of transmission or pedagogical tradition that diverges from the Arabic scholarly canon.

A square quadrilateral that you take away from the number of its area the number of its four sides, and [you] are left with 21 cubits of its area: what is the area and what is the number of each side of the square?

Answer: Divide the number of the sides, which is four, into two. Multiply the two by itself, which is 4. Add this number to the given number that's left over from the square, and the total is 25. Find the root of 25, which is 5. Add half the sides, which is 2, so the total is 7. This is the side of the square, and its area is 49. He who posed the question subtracted from the area, which is 49, the number of the four sides, each of which is 7 and all four 28, leaving from the square 21, as he told you.

If you want the proof that your answer to him is correct, let this square be *ABCD* with equal sides. It is known that each side has more than four cubits, because he who posed the question said that we subtracted from the square all four sides [considered as a rectangle that has] 4 cubits in width and in length equal to the length of the square, leaving so and so;[32] if the side had no more than 4, he could not have subtracted 4 sides. We therefore subtract from the line *AB* a line whose length has 4, which is the line *BE*, and from the opposite line *CD* a line whose length has 4, which is the line *CG* [Fig. III-2-2]. We make a line from the point *E* to the point *G*, and then divide the line *BE* into two equal parts at the point *H*. The lines *BH* and *HE* are equal, and each has 2 cubits in length. You see in this square that the rectangle marked by *BEGC* is the four sides of the square joined together, as this rectangle is the line *BC* times the line *BE*, which is four, and the line *BC* is the side of the square; we count it 4 times, which is the number of the four sides. So you see that the rectangle *BCEG* is the four sides of the square. When you subtract it from the square, there remains the rectangle *ADEG*, which is known to be 21, as the number left to [him] who posed the question.

Now observe that the line marked *BE* is divided into two equal parts *BH* and *HE*, together with another line, *EA*. But we've already told you[33] that for any line divided into two equal parts and joined with another line, the rectangle of the entire line with the joined line by the joined line, and the square of half the first line, added together, equal the square of half the line with the joined line both together by themselves. Therefore the rectangle of the line *BA* by the line *AE* and the square of the line *EH* by itself together equal the square of the line *HA*. The rectangle of the line *AB* by the line *EA* is the rectangle *ADEG*, because its one side *AD* equals the side of the square and its other side is *EA*, which is

[32] Note that while the question is set in terms of subtracting numbers, the solution turns the subtracted 4 sides into a rectangle whose sides are 4 and the side of the given square.

[33] Bar Ḥiyya's Book I, §29, equivalent to *Elements* II.6.

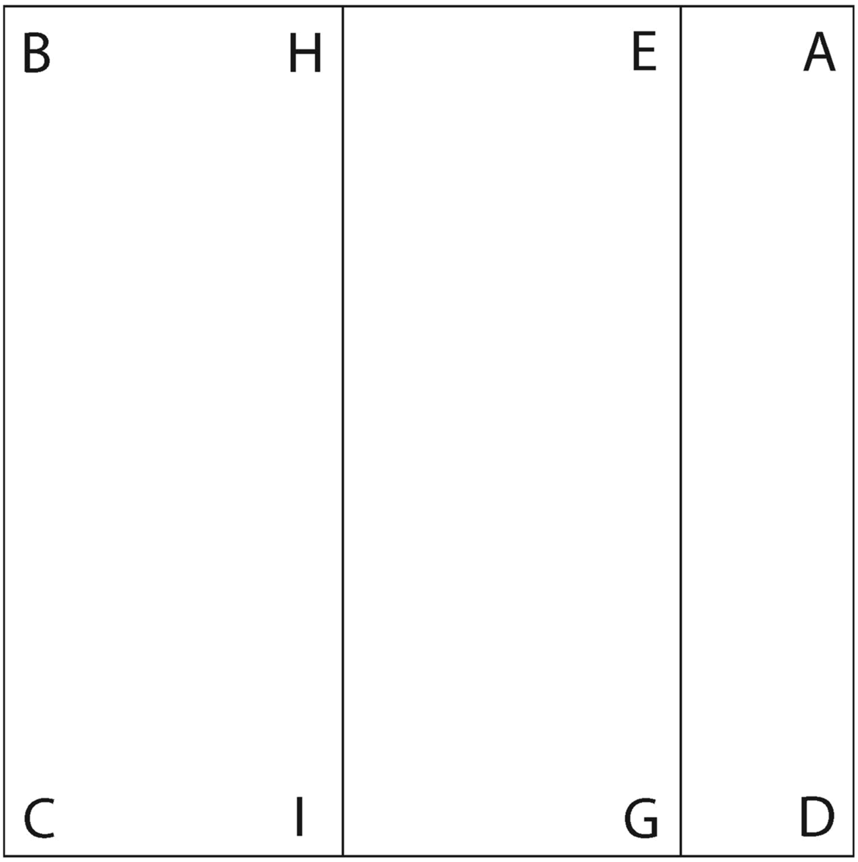


Fig. III-2-2.

the line joined with the line *EB*. This rectangle is 21, and if you add to it the square of the line *EH*, which I know to be 4, the total will be 25. Its root is 5, and it equals the square of *HA*. You see that the line *HA* is the root of the square 25. Add *BH*, which is 2 cubits, making a total of 7, which is the line *BA*. And so the rectangle *ABCD* is 49 cubits, as I showed you in this diagram.

If he says: “A square that you add the number of its four sides to the number of its area, making in total 77, how much is this rectangle?”

In this question take half the number of the sides, which is two. Multiply it by itself, making in total 4. Add this to the number that he gave you, 77, making 81. Take the root of this number, which is 9. Subtract from it half the number of sides that you added, and you are left with 7. This is the side of the square, and its area is 49.

[Translation of the proof is omitted.]

A square that you take away its area from the number of its four sides, and are left with three.

In the answer to this question you divide the number of the sides in two. Their half is 2 and their square is 4. Take away the three that you were left with, leaving 1, whose root is 1. Subtract it from half the sides, and you are left with 1, which is the side of the square, or add the root of the 1 you are left with to half the sides, making 3, which are also the side of the square. It can be one and can be 3, as this question has two solutions

[Translation of the proof is omitted.]

Measurements of rectangles (from Book II)

This example shows how quadratic problems are solved by reduction to elementary geometric identities (typically from book II of the *Elements*, which may be interpreted here as a form of "geometric algebra"), rather than to canonical quadratic equations. Note also the terms of the problem: the givens are the difference of two numbers and the root of the sum of their squares, and the numbers are sought. This is then reduced to the case of given products and differences. This way of posing problems echoes pre-Arabic geometric-algebraic traditions.

A non-square rectangle whose diagonal has ten cubits and its length exceeds its width by two: how much is its length, width and area?

Answer: You know that the square of the diagonal is a hundred. Take away the square of the excess of the length over the width, which is 2, and its square 4. You're left with 96 of the hundred. Divide it into two, making 48, which will be the area of the rectangle [this is justified by reference to a previous section]. If you want to know its sides, of which one adds two to the other, divide this excess into two, making one, and its square one. Add this to the area, making 49, whose root is 7. If you add one, which is half the excess, it will be 8, which is the length line. If you subtract one, 6 will be left, which is the width line. 8 times 6 is 48, which is the area.

The proof for this matter: Let the vertices of a non-square rectangle be marked *ABCD* and its diagonal *AD*, which we set as 10 cubits [Fig. III-2-3]. We know that the line *AB*, which is the length, adds to the line *AC*, which is the width, 2 cubits. And we want to know from these two numbers the area of the rectangle and the magnitude of each of its sides. It is known that the rectangle of the line *AB* by the line *AC* is the area, and that the square of the diagonal equals double the area together with the square of the excess of the length over the width, as I showed you above.[34] Therefore, if you subtract from the square of the diagonal, which is 100, the square of the excess 2, whose square is 4, there remain 96, which is twice the area. Its half is 48, which is the area.

If you want to know the numbers of the sides, you already know that the length adds 2 to the width. Subtract now from the line *AB*, which is the length, a line equal to the line *AC*, which is the width. This line is *BE*, leaving the line *EA*, which is known to be 2, as the excess of the length over the width. If you bisect the line *EA* into two at the point *G*, the line *EG* and the line *GA* will be one cubit each. So *EA* is divided into two equal parts at the point *G*, and you add another line, *BE*. You know that the rectangle of the entire line *AB*, which is the line with the addition, by the line *BE*, which is the added line, together with the square of *GE*, which is the half, equals the square of *GB*, which is the half with the addition.[35] The rectangle of the line *AB* by the line *BE* is the rectangle *ABCD*, which is 48 cubits. Since the line *EB* equals the line *AC*, which is the width, and the square of *EG* is one, added together they are 49 and equal the square of *GB*. Therefore the line *GB* is

[34] Bar Ḥiyya's Book I, §30 = *Elements* II.7.
[35] Bar Ḥiyya's Book I, §29 = *Elements* II.6.

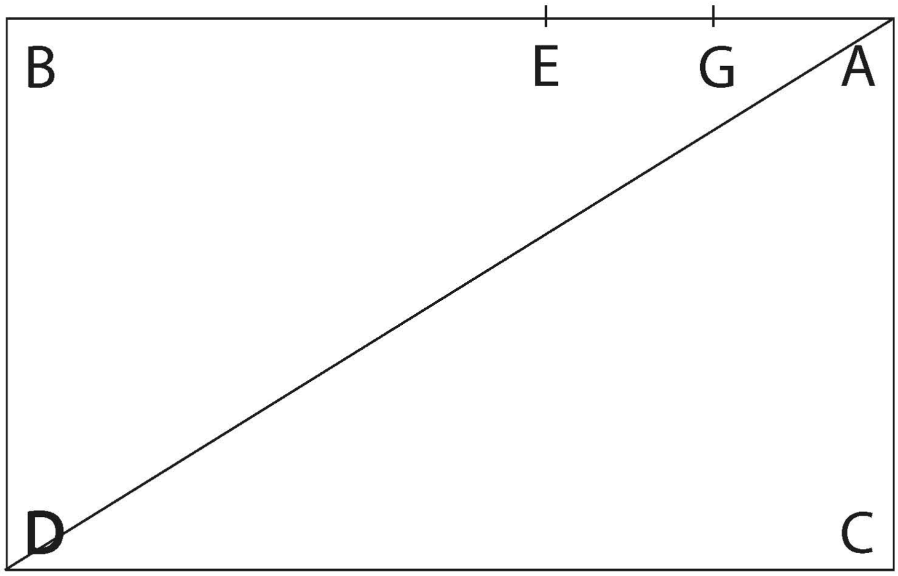


Fig. III-2-3.

7 as the root of the rectangle 49. If you add *AG*, which is one, the entire line *AB* is eight, which is the length of the rectangle. If you subtract from it this line, which is again one, there remain 6 for the line *EB*, and this line equals the width of the rectangle, which is the line *AC*. The rectangle of the length by the width is the area as in this diagram.

In this next example, the solution depends on circle proportion theory, rather than on book II of the *Elements* interpreted as "geometric algebra."

A non-square rectangle whose diagonal together with its side is 18, and its other side 6: how much is its area, and diagonal, and the side added to the diagonal?

He who answers this question will take the square of the known side 6, whose square is 36. He will then divide them by the diagonal and the side, which are 18, making 2. He will add 2 to 18, making 20. It is known that half of 20 is the diagonal, which is 10, and what remains from 18 [after subtracting 10] is the side added to the diagonal, which is 8. The rectangle of 8 by 6 is the area, which is 48.

The proof for this answer is this. Let this rectangle be the rectangle *ABCD*, and let its diagonal be *AC* and the unknown side *AD* and the known *CD*. Set the point *A* as a pivot, and make a circle with a compass at distance *AC*. This is the circle marked *ECG*. Extend the line *AD* to the circumference of the circle both ways to the point *E* and the point *G* [Fig. III-2-4]. You have the line *AE* equal to the line AC, which is the diagonal, because they both set out from the pivot called [in Arabic] *markaz* to the circumference. He who posed the question set the line *AC* together with the line *AD* as 18, so the entire line *ED* is also 18 and the entire line *EG* is the diameter of the circle. And it is known, as I expounded among the established reasoning,[36] that the rectangle of the line *ED* by the line *DG*, which

[36]Bar Ḥiyya's Book I, §33 = *Elements* III.35.

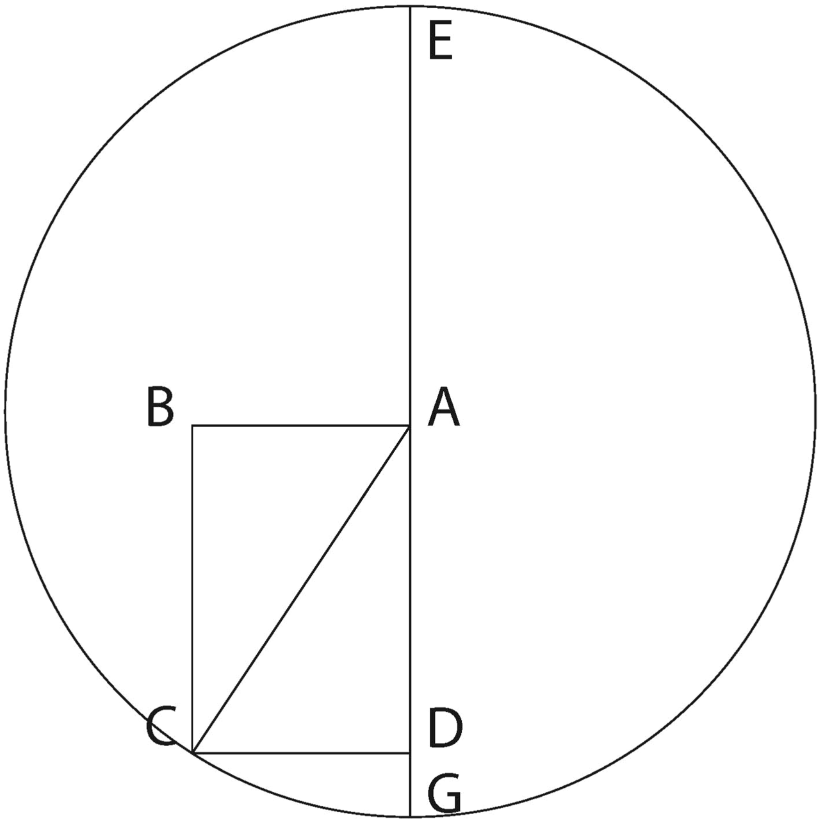


Fig. III-2-4.

complements the diameter, equals the square of the line *DC* by itself, because these are two lines in the same circle dividing each other, and one of them goes through the pivot of the circle. Therefore, if you take the square of the line *DC*, which is six, by itself, and we divide it by the line *ED*, the outcome is the line *DG*, which is 2. The whole line *EG* is 20, and half the line *EG* is the line *AE*, which equals the diagonal. So it is ten, and there remains 8 for the unknown line *AD*, as we answered you, and as seen in this diagram.

Measurements of triangles (from Book II)

Here an extension of the Pythagorean theorem to general scalene triangles is used to determine the height of a triangle from its sides, which in turn would serve to find its area.

If, in the triangle *ABC* which we gave you [with sides 13, 14, and 15], we wish to take the height between the sides *AB* and *AC* onto the base *BC* of length 14 cubits, we should first find the part of the base to one side of the height [Fig. III-2-5]. To extract the longer part of the base, take the square of the longer of the two sides surrounding the top angle of the triangle, between which we take the height. This is the side *AC* of length 15 cubits. We add to this square the square of the base. These two squares together are 421. We take away the square of the remaining side *AB*, which is the short side, whose square is

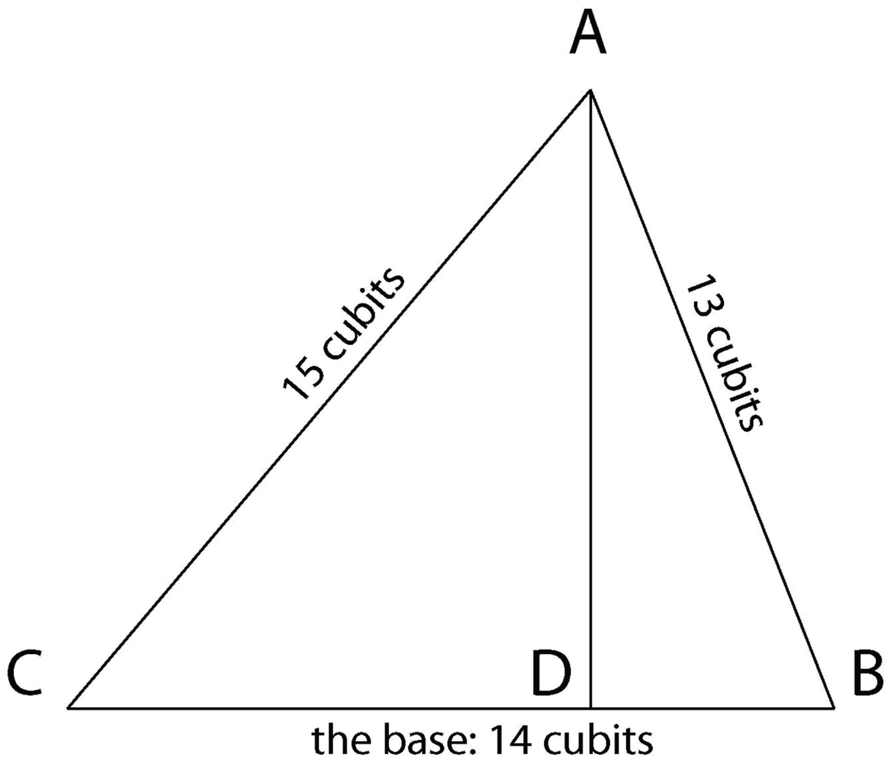


Fig. III-2-5.

169, leaving you with 252. We divide the remainder in two. Its half is 126. Divide this half by the base, which is 14, and the result is 9, which is the distance along the base from the height to the long side.

. . .

We operate [in] this way for every side for which we want to calculate the height. And once we know the part of the base to one side of the height, we come to know the length of the height thus: we square the side, take away the square of the adjacent part of the base, and take the root of the remainder, which is the length of the height.

. . .

If you ask for the proof for the calculation of the height, look at the diagram of this rectangle, which I draw for you now [see Fig. III-2-5]. Know that the square of the side *AB*, which is opposite to an acute angle, as in this triangle, is less than the square of the side *AC* and [the square of] the side *BC*, which is the base; the excess is double the rectangle of *CD*, which is one part, by the entire base *BC*, as taught in geometry.[37] When you divide this excess in two and this half by *BC*, you get *CD*.

[37] *Elements* II.13, from Bar Ḥiyya's Book I, §27.

Heron's rule for determining the area of a triangle from its sides is brought here as well, without proof (and without reference to Heron).

Although [the calculation of the area of a triangle] is as I said, and all the ways that I showed you are correct and clear to he who understands, and yield a true result, you can find a method for calculating and measuring triangles that does not require the height, namely, the so-called calculation by excesses.

This method here requires that you find half of each of the sides of the triangle, and sum these halves together, and find the excess of the sum over each side, and note down these excesses. Then multiply one of them by another, and multiply the product by the third excess, and multiply what you get from this calculation by the sum of the halves that you added together. This number is the square of the area of the triangle. If you take the root of this number, you will find the area.

The author then uses this method to calculate the area of a triangle with sides of length 10, 8, and 6.

This calculation is based on the principles of geometry, and its proof is taught there. We cannot state it here, and we do not have much need for it here, because you saw that this calculation is true by the numbers that I gave you.

Circle and arc measurements (from Book II)

This original presentation of the reduction of the area of a circle to that of a triangle is probably the most famous section of the book. In violation of the classical (Aristotelian) tradition, it depends on decomposing an area into the lines that it contains.

Once we know the circumference and diameter, we know the area of the whole circle, which is half the diameter times half the circumference. The proof for this area: We know that if you open the area of the circle on one side, and straighten all the surrounding lines from the external line [circumference] to the center, the lines surrounding the area of the circle will spread and turn into straight lines, decreasing until they turn into a single point, which is the center point. Such is the line *ABCD* . . . that I have drawn, where the external is the largest, and the next is smaller than the former but larger than the next, and so on to a point, which creates the form of a triangle [Fig. III-2-6]. But we have already taught the area of a triangle, which is as the height times half the base, which is half the diameter times half the circumference.

This section presents a table that calculates the arc from the length of a chord. The choice of the full chord, rather than the half chord (Sine) is typical of the Greek, rather than Indian or Arabic traditions. The normalization here is rather idiosyncratic: the diameter is counted as 28 parts, yielding a circumference consisting of 88 such parts.[38] The reconstruction of Table III-2-1 presented below is somewhat conjectural, as the manuscripts are obviously corrupted by scribal errors and often diverge. We also include an example for the use of the table, which depends on rescaling the measurement so that the given diameter fits the table's value of 28.

Suppose you know the diameter of one circle, and you are given a chord, and want to know the length of the arc surrounding that chord. You seek a rule that lets you find the length of the arc from the sagitta or chord, like the rule for finding the length of the diameter

[38]This value is used because 88 is $3\frac{1}{7}$ times 28. Given the approximation $\pi = 3\frac{1}{7}$, this guarantees that the parts of the diameter and circumference are equal.

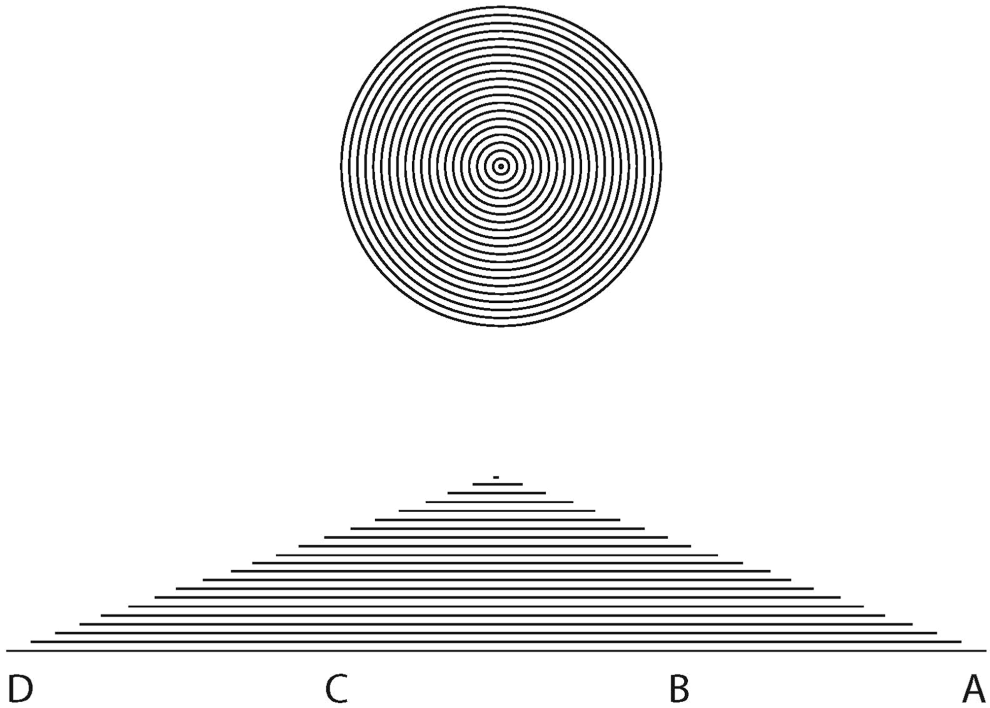


Fig. III-2-6.

from the circumference and the length of the circumference from the diameter. Know that such a rule cannot be given to you, because the ratio between the chord and the arc is not fixed, but changes with the changing arcs and chords. . . . As a result, the reckoning of arcs and chords is difficult for most people. He who reckons it must understand many rules of geometry.

The scholars of astronomy labored on this for their practice, and I copied from this reckoning what I saw fit for this book. I drew for you a table divided lengthwise into 28 parts, as I divided the diameter of the circle into 28 parts. And according to this the circumference is divided into 88 parts. I divided the table along its width into 4 columns, and noted in the first of these columns lengthwise the 28 parts, and in the remaining 3 columns along the width I noted lengthwise the arc fitting each chord from 1 to 28. This table of arcs is divided into 3 columns because I divided each part into 60 seconds [sic], and each of the seconds I divided into 60 thirds, as you see in this table [Table III-2-1]. . . .

An example for this calculation. In a circle whose diameter contains 10 and a half, we take a chord whose length is six and ask to know the length of the round arc over that chord. We multiply the length of the given chord, which is 6, by 28, which is the diameter in the table. We get 168. We divide this number by 10 and a half, which is the diameter of the given circle. The result is 16, which [with respect to the table's diameter] is the chord in the proportion of your given chord to the given diameter. Against it you find in the table 17 large parts and 2 seconds and 16 thirds, which is the arc appropriate for the table's chord. Now multiply again this arc by 6, which is the number of the given chord. The total is 102 large parts, 13 seconds and 36 thirds. Divide this number by 16, which

Table III-2-1

Chord	*Arc*		
	Parts	*Seconds*	*Thirds*
1	1	0	3
2	2	0	8
3	3	0	25
4	4	0	55
5	5	1	44
6	6	2	57
7	7	4	42
8	8	7	1
9	9	9	59
10	10	13	42
11	11	18	33
12	12	24	23
13	13	31	29
14	14	40	0
15	15	50	10
16	17	2	16
17	18	16	36
18	19	33	29
19	20	53	29
20	22	17	10
21	23	45	19
22	25	19	4
23	27	0	1
24	28	50	36
25	30	54	52
26	33	20	55
27	36	29	29
28	44	0	0

are the table's chord, and it will be 6 large parts, 23 seconds and 21 thirds, which is the measure of the required arc.

Sloping terrain measurement (from Book II)

The area of a parcel of land on a sloping or hilly terrain should be measured, according to this text, as if it were "projected" on an underlying plane. The reason is that crops and structures rise up vertically, and therefore the area must be measured only with respect to the vertical rise it enables. In the case of a hill shaped like a circular arc, the chord-arc table is called for. Although anachronistic, I cannot help but see in the image of a sloping area reckoned with respect to a vertical field of crops the modern integral of an area form on a manifold with respect to a vector field. A crucial difference is that the text considers the area globally, rather than breaking it up into small (or infinitesimal) local patches.

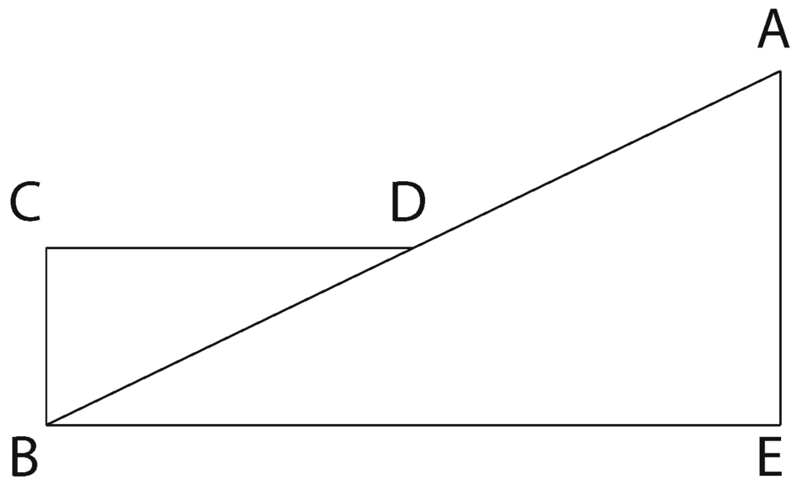


Fig. III-2-7.

All I taught you so far concerns measuring flat lands, where the terrain is spread straight without climbing up or down. But sometimes you come by terrains sloping down from the head of the mountain, or submerged low, or round and curved. The surveyors in these lands are wrong to measure all terrains, high or low, in the same way. You take care, and if you come by a lot [*AB* in Fig. III-2-7] hanging from the head of a mountain, find its height [*AE*], which is the distance from its beginning to its high end, and subtract the square of the height from the square of the length of the lot; the root of the remaining number times the width of the lot is its area.

If the lot falls low, treat its downward fall in the same way. And for curves on top of the hill, seek and find a way to take its area according to the straight plane on which it sits. Indeed, neither seed nor building rise but according to a straight angle with respect to the straight land, and the excess measured in a high or low land is useless for both seeds and buildings. Therefore you must subtract it and set it against the straight measurement of the plane terrain.

Those clever in measuring land sought to know the height of lands sloping off mountains and hills in order to extract the correct measure of the plane terrain whose area they are to find. They would do thus: They would erect a pole [*BC*] on the lower part of the terrain at a right angle with respect to the plane, and set at the head of that pole another pole [*CD*] at a right angle, and extend this pole set against the other, and lengthen it until it would reach the terrain sloping off the mountain wherever it falls. Then they would measure from the bottom of the pole at the lower part of the terrain to where the other pole reached higher up [*BD*]. This number is always found to exceed that pole which extends from the top of the standing pole to where it touches the ground. They would then use the ratio of the excess to reckon the excess of the area of the sloping terrain with respect to its area if the land were plane.

. . .

Suppose the terrain slopes along a round and curved hill, and you wish to find its true area, fitting the flat lot on which the land stands. Erect the first pole [*BC*] at a right angle

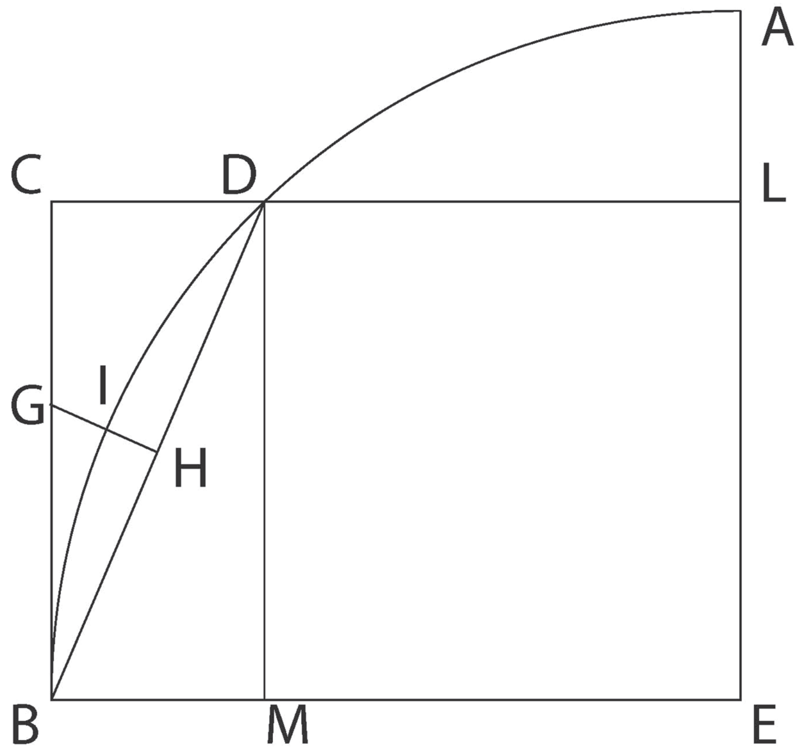


Fig. III-2-8.

to the plane land, and set on top of it another pole [*CD*] a right angle, and extend it until it reaches the curved lot as I showed you for the terrain sloping straight from the head of a mountain [Fig. III-2-8]. You will get the same diagram, except that the line *DB* is a round arc, and so is the entire *AB*, which is the arched terrain, both arcs being part of the same circle. Both arcs are known, as the arc *AB* is the length or width of the terrain you measure, and the arc *DB* is the arc surrounded by the two lines [poles]. You can find the chord of the arc *DB* from the erected triangle *DCB* by squaring the lines *DC* and *CB* surrounding the right angle C of the triangle *DCB*, adding the two squares, and taking the root of their sum, which is the length of the chord *DB*.

Now that we know this chord we can reckon to find the sagitta and from that the diameter. It is known that the sagitta divides the arc and the chord into two equal parts. We let *HIG* set out at a right angle from the point H at the middle of the chord, dividing the arc *DB* in the middle at the point I, and extending until it reaches the line *CB* at the point G. So we have the triangle *GBH* similar to the triangle *BCD*, and the ratio of *DC* to *CB* as the ratio of *GH* to *HB*. Now *DC*, *CB*, *HB* are all known, and the line *GH* is unknown. If we multiply the line *DC* on one side of the ratio [*muqash*] by the line *BH* on the other side of the ratio [*noqesh*], which are both known, and divide their product by *CB*, which is known, the result is the length of the line *GH*.

From this ratio we come to know the length of the line *HI*, which is the sagitta. Indeed, the ratio of *DC* to *DB* is as the ratio of *GH* to *GB*, and if we multiply *GH* by *DB*, which

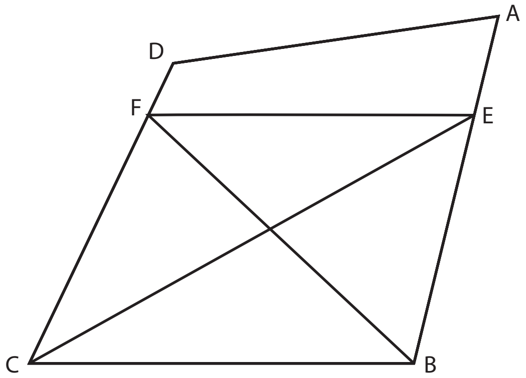


Fig. III-2-9.

are both known, and divide the product by the known *CD*, we find the length of *GB*. We measure from the known point *G* the length of *GI*, and subtract from the entire line *GIH*, leaving us with the length of *HI*, which is the sought sagitta. And from this sagitta and the chord *DB* we can find the diameter of the circle as we taught above [using the proportion of the parts of intersecting chords]. Having found the diameter, we can find the chord of double the arc *AD*, as we taught in the table of arcs and chords. Half this chord is as the line *DL*, and if we add to it the known line *DC*, the entire line *CDL* becomes known and is equal to the line *BME*, which should replace the curved line as in this diagram.

Division of areas (from Book III)

Here are two of the more clever divisions of a plane area into halves, borrowed indirectly from Euclid's book *On Divisions*. The first division divides a quadrilateral from an arbitrary point on a side, while the second divides a region partially bounded by a circular arc. The Greek text had disappeared by Bar Ḥiyya's time, but an Arabic abstract of the work by the tenth-century Persian geometer al-Sijzī, including the theorems and a few proofs, may well have been available to him [Hogendijk, 1993]. Compare these to Fibonacci's divisions in section II-4-3 of Chapter 1.

The first construction shows how to divide a quadrilateral in half, where no diagonal divides it in half, starting from a given point on one of the sides.

A different method of division [is effected] by drawing a partition from a point E on the side AB in this irregular quadrilateral shape, as in the diagram I draw [Fig. III-2-9]. First, we divide the quadrilateral [into two equal parts] from the vertex B on the line AB, as you were instructed in the previous diagram [not included here]. The parts are the triangle BCF and the quadrilateral ABDF, into which the quadrilateral is divided by the line BF drawn from the point B. We draw a line from point F to point E. If this line is parallel to the line CB, we draw a line from point E to point C, and the quadrilateral is divided in half along the line EC. The quadrilateral AECD equals triangle EBC as in the first diagram for this problem [Fig. III-2-9].

The proof of the division: these are two parallel lines, line EF being parallel to line BC. The entire triangle BCF, which is half of the entire quadrilateral, is equal to triangle BCE, both lying between two parallel lines. Therefore, triangle EBC is half the quadrilateral.

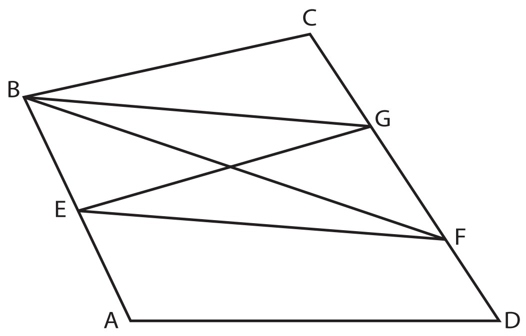


Fig. III-2-10.

Suppose the line EF is not parallel to the line BC. We draw from point B at the angle of the quadrilateral the line BG parallel to the line EF. It is either interior to or exterior to the quadrilateral. Suppose that it is interior, as in the second diagram [Fig. III-2-10]. We draw a line from point E to point G. It divides the quadrilateral in half, namely, the quadrilaterals AEDG, EBCG. This is known because, as we noted above, triangle FBC is half the quadrilateral, and triangle BGF is equal to triangle BGE, as they lie on the same base between parallel lines. If one adds to each the triangle BGC, then triangle FBC will become equal to quadrilateral EBCG, and therefore quadrilateral EBCG is half of the entire quadrilateral. The other half, as we noted in this second diagram, is quadrilateral AEDG.

Suppose the line falls outside the quadrilateral, as line BG in the third diagram [Fig. III-2-11]. We extend the line CD to the point G, draw lines from E to G and from E to C, draw from G the line GH parallel to the line EC, and then a line from E to H. The quadrilateral is thereby divided into two equal parts, namely the triangle EHB and the pentagon AEHCD.

The proof for this division: Since line EF is parallel to line BG, triangle BFG is equal to triangle BGE. Similarly, triangle GEH will equal triangle CGH, because they lie between the parallel lines CE and GH. If we remove the triangle CGH from triangle BFG, and the triangle GEH from triangle BGE, you will have triangle BGH common to both [remainders]. Therefore remove from these remainders triangle BHE, and there will remain the triangle BCF equal to triangle BHG. The triangle BCF is half of the quadrilateral, and so is triangle BEH as in the third diagram.

You can draw a line from point G parallel to line CE in this third diagram, if you know the ratio of GC to CF, and if you extract, according to the same ratio, LM from the extension of line FL toward B. Draw the line GHM from point G to point M. This line is parallel to line CE, since if you draw the line CL in the triangle FMG, the ratio of CG to CF equals the ratio of LM to LF. Thus the line GH is parallel to the line CE, as we have told you.

In the next construction, the author divides a region partly bounded by a circular arc.

Suppose you have a portion with one round line and the other sides straight, as the portion *ABC* that I draw for you, where the two sides *AB* and *BC* are on straight lines and the side *AC* is somewhat round. This is called a truncated portion. If you wish to cut this shape into two [equal] parts, draw a straight line *AC* and divide it in the middle at the point *E*. From there draw a height on the line *AC* reaching the arc *CA* at the point *G*. Draw a line from *B* to *E* and this truncate is split into two equal parts along the line *BEG*, if it goes

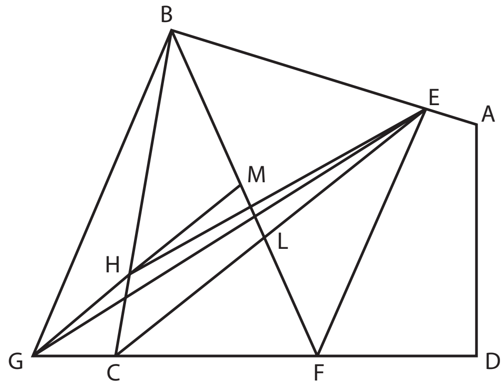


Fig. III-2-11.

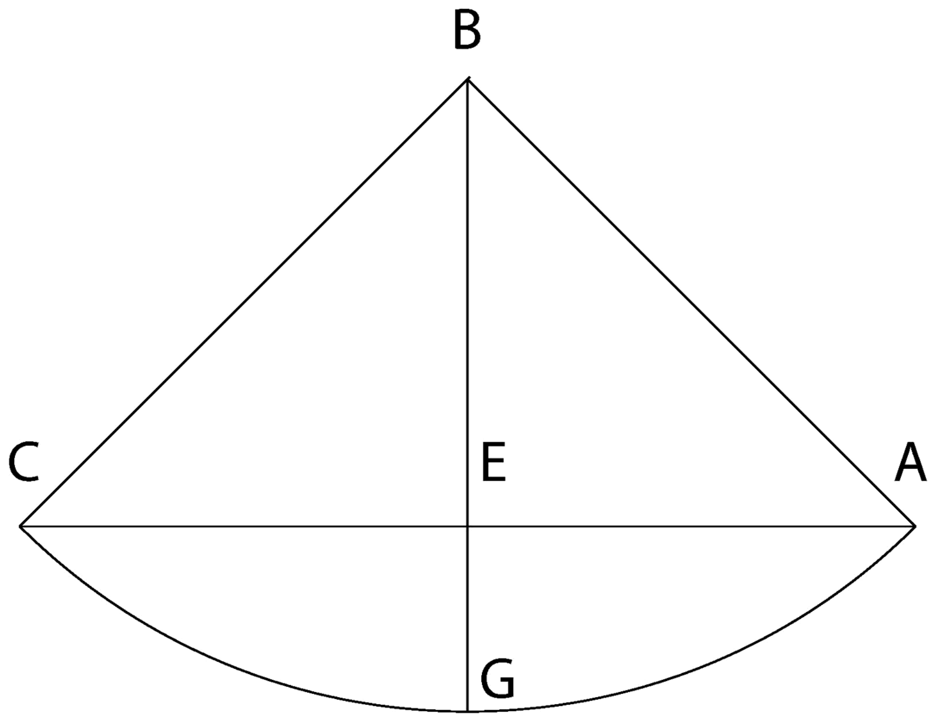


Fig. III-2-12.

along one straight line as in the first diagram [Fig. III-2-12], or along the lines *BE*, *EG* if they are not on a straight line, as in the other diagram [Fig. III-2-13].

You can also divide the second diagram in another way. Draw a line from *B* to *G*, which crosses *AEIC* at the point *I*. Find the ratio of *IE* to *EA*, and divide the line *AB* [at *H*, so the parts are in the same ratio] as the ratio of *EI* to *EA*. We draw a line from *G* to *H*, and the truncate splits into 2 equal parts, which are the truncate *GHA* and the irregular *BHCG* as in this diagram [Fig. III-2-14].

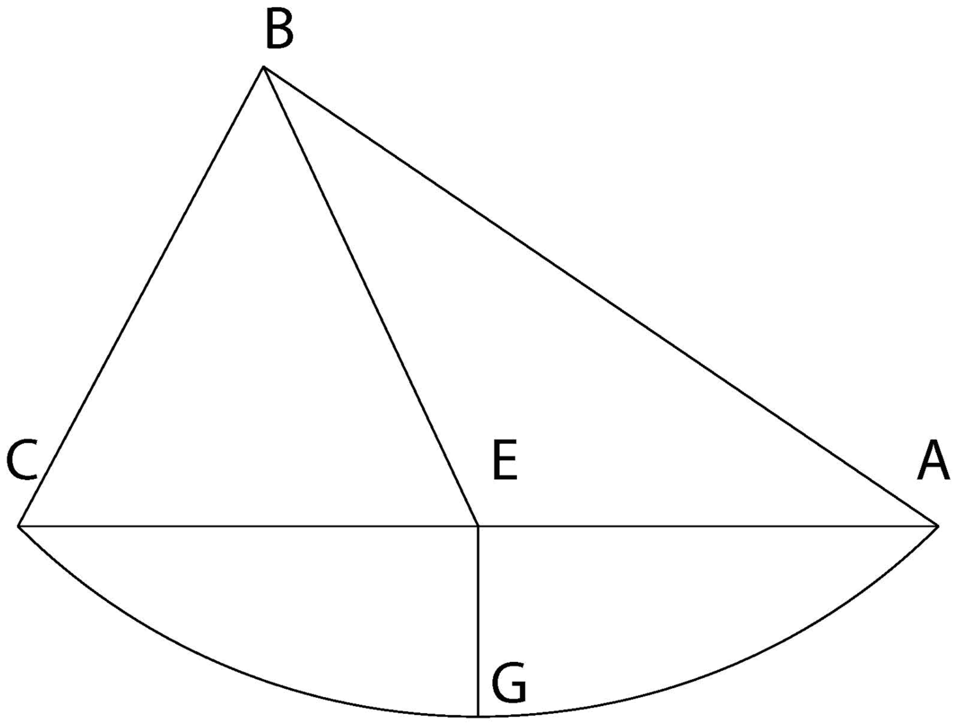


Fig. III-2-13.

The proof for this diagram is that the triangle GHE and the triangle BHE, which are between two parallel lines, are equal. So the truncate *AHG* equals the triangle *ABE* together with the truncate *AEG*, which are both half the truncate *ABC*, as in this diagram.

As we saw in Bar Ḥiyya's introduction, some of the motivation for the Hebrew literature on measurement was religious. Here we include an early example from Rashi, which predates Bar Ḥiyya, and a later example, by Simon ben Ṣemaḥ, which builds on Bar Ḥiyya's work.

3. RABBI SHLOMO ISHAQI (RASHI), *ON THE MEASUREMENTS OF THE TABERNACLE COURT*

This section was prepared by David Garber

The following comment is by Rashi (acronym of Rabbi Shlomo Iṣḥaqi, 1040–1105). Rashi spent his life in Troyes, except for a decade of studies in Mainz and Worms. His importance among Jewish exegetes cannot be exaggerated, no doubt supported by his clear, succinct, and literal style. His commentary on the Talmud is a standard fixture in Talmud editions.

The commentary here applies elementary geometric considerations to the context of the temple measurements as discussed in the *Mishna*, deploying a cut-and-paste procedure for squaring a rectangle. Albeit elementary, the presentation is clever and elegant.

Rashi commentary to: "The length of the [tabernacle] court shall be a hundred cubits, and the breadth fifty everywhere, the Torah having thus ordained: 'Take away fifty and surround [with them the other] fifty.'" (*Eruvin* 23b)

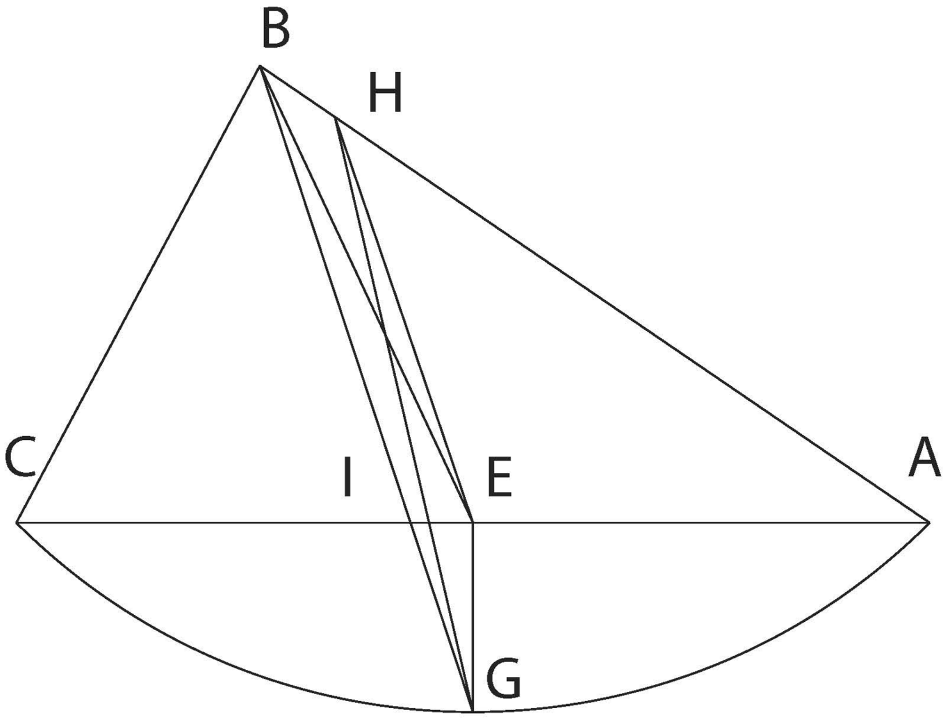


Fig. III-2-14.

"Take away fifty," which is the excess of the length over the width, and use them to surround the remaining fifty so as to form [a square of] seventy cubits and 4 palms [*tefahim*]. How so? Split [the remaining fifty by fifty] into five strips of ten cubits breadth and fifty cubits length [Fig. III-3-1]. Set one strip to the east and one to the west. Then the breadth is seventy and the length is fifty. Next set one to the north and one to the south. Then you have seventy by seventy, but the corners are missing ten by ten each with respect to the addition you made. Take from the remaining fifty 4 pieces of 10 by 10, and set in the four corners, so they become complete. There remains one strip of ten by ten cubits left, which is sixty palms by sixty palms. Split them into 30 strips of two palms (then you have 30 strips, each ten cubits in length). Altogether they add up to 3 hundred cubits. Set 70 [cubits, or 7 strips] in each direction, then you have seventy and 4 palms by seventy and 4 palms, but the corners are missing two palms by two palms. You have twenty cubits left. Take eight palms and set them in the corners, so they become complete. You have 18 cubits and 4 palms left of two palms width, that is, but a trifle, which, if you try to split to surround the square, the excess would not reach two thirds of a finger in breadth [Fig. III-3-2].[39]

[39]Note that the procedure can be continued indefinitely, but that Rashi shows no infinitesimal inclinations here and terminates the calculation when the error becomes small enough for the given context.

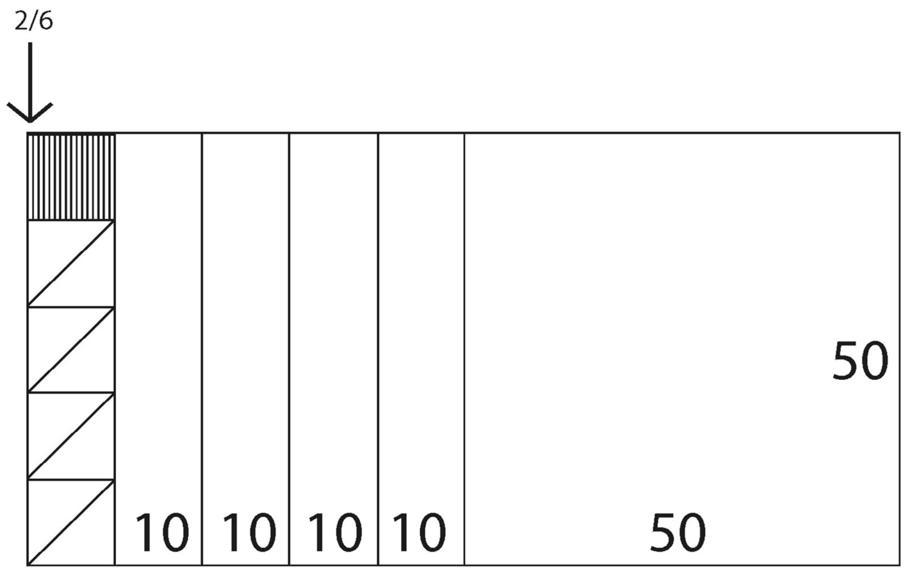


Fig. III-3-1. The rectangular court.

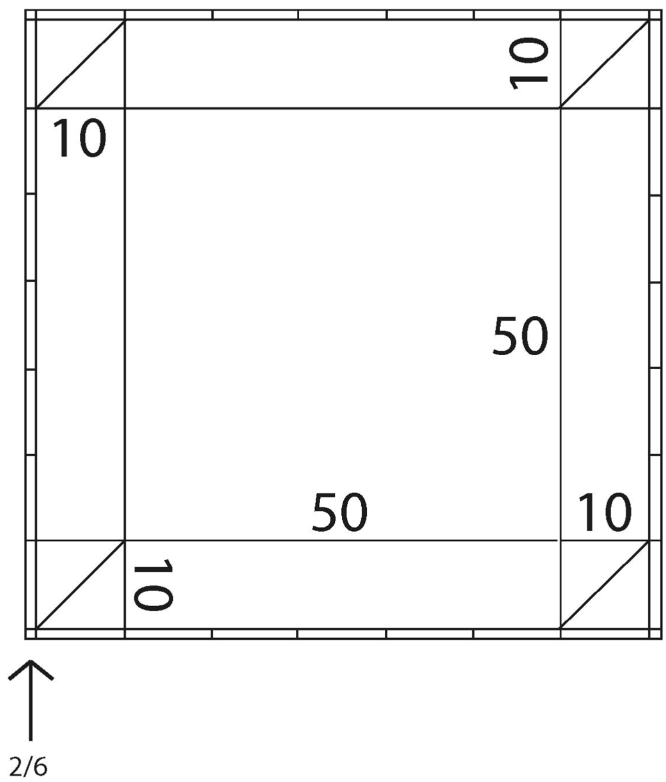


Fig. III-3-2. The squared court.

4. SIMON BEN ṢEMAḤ, *RESPONSA ON SOLOMON'S SEA*

This section was prepared by David Garber

Rabbi Simon ben Ṣemaḥ Duran (1361 Mallorca–1444 Algiers) was one of the greatest rabbinical authorities of his time. He made his living as a physician in Aragon but had to flee to Algiers due to the 1391 riots against local Jews. In Algiers he served in the tribunal of Rabbi Isaac ben Sheshet. Ben Ṣemaḥ's responsa book, the *Tashbeṣ* (the acronym for the responsa of Simon ben Ṣemaḥ), deals with about 800 questions and has four parts: three written by himself and one by other rabbis in his family. The book includes several discussions of mathematical aspects of religious law.

This excerpt from question 165 of the first part of the *Tashbeṣ* includes a discussion between Rabbi Simon ben Ṣemaḥ and Enbellshom Efrayim (perhaps the Mallorcan astronomer and brother of Ben Ṣemaḥ's teacher, Vidal Efraim; "En" is an agglutinative Catalan honorary prefix, equivalent to the Castilian "Don"). The interpretation of the discussion follows [Garber and Tsaban, 1998].

Enbellshom, following Bar Ḥiyya (but without mentioning him as his source; see [Garber and Tsaban, n.d.] for a comparison), claims that religious authorities knew that the value $\pi = 3$, used for religious calculations, was only an approximation, and, moreover, that they were aware of more exact approximations, such as $\pi = 3\frac{1}{7}$, which was known at the time. According to this view, the Talmudic scholars only used the coarser approximation where it rendered religious rules more strict. This assumption, however, calls for a reinterpretation of various Talmudic debates.

In contrast, Ben Ṣemaḥ, while upholding that religious authorities were no less savvy than Euclid and Archimedes, claims that their calculations sometimes depended on the approximation $\pi = 3$ even if it rendered religious rules more lenient. Nevertheless, he qualifies the injunction to adhere to precise calculations. Ben Ṣemaḥ seems to reconcile the two positions by recalling the instability of measurement units and the authorities' attempts to render religious law accessible to laymen. The underlying concerns of the debate are the place of scientific knowledge in the interpretation of religious texts and the limits of exegetic practices.

In the excerpt presented here, the topic discussed is the measurement of Solomon's Sea, a large ceremonial basin in King Solomon's temple. In Kings I, 7:23, the Sea is stated to be 10 cubits in diameter, 5 cubits high, 30 cubits in circumference, and 2,000 *bat*s in volume (which equal 6,000 *seʾa*s or 450 cubic cubits or 150 kosher ceremonial purification basins (*miqve*)). Since the volume of such a cylinder, assuming $\pi = 3$, should only be 375 cubic cubits, the Talmud suggests that the three bottom cubits were square, and only the top two were circular, yielding the required volume of 450 cubic cubits. Enbellshom quoted Bar Ḥiyya's alternative estimate of the measures of Solomon's Sea (without reference to Bar Ḥiyya, as mentioned above). This revision was rejected by Ben Ṣemaḥ on the grounds stated below.

First we present Bar Ḥiyya's argument, as quoted from Enbellshom by Ben Ṣemaḥ.[40] In this argument, the scriptures are reinterpreted so as to reduce the diameter of the circular part of the Sea to cohere with the prescribed volume, circumference, and $\pi = 3\frac{1}{7}$. Then, assuming that the size of a *miqve* is defined as one part in 150 of Solomon's Sea, it is shown that a

[40] Levi ben Gershon also considered this issue of the exact dimensions of Solomon's Sea, but there is no evidence that Ben Ṣemaḥ was aware of Levi's work. See [Simonson, 2000a, pp. 7–8].

kosher *miqve* can be slightly smaller than the standard prescription of 3 cubic cubits. After discussing the supposed true measurements of the Sea, it is explained how a doubt concerning the interpretation of the thickness of the rim of the Sea led the Talmudic scholars to reach a different volume estimate, even though, like him, they supposedly assumed $\pi = 3\frac{1}{7}$.

It is written that [Solomon's Sea] is 10 cubits from side to side, circular in form, 5 cubits high, with circumference 30 cubits round. It seems at the beginning that this neglects the seventh [in $\pi = 3\frac{1}{7}$]. But here things are precise as well, if one looks carefully, for it says that "it was a palm thick, and its brim was made like that of a cup, like the petals of a lily."[41] It appears that the circular part is one palm away from the edge of the square, which makes two palms in diameter out of 5 palms per cubit. . . . If you multiply 9 cubits and three fifths [= 10 cubits less 2 palms] by three and a seventh, the external circumference of the Sea will be 30 cubits and a sixth approximately.[42] Its inner circumference should be 30 cubits, so the thickness of the rim, which is like a flower, is about two thirds of a finger.[43]

If you multiply half the diameter by half the circumference correctly, and consider the width, you find that the area of the circle is 72 cubits and 2 ninths.[44] This is the volume per one cubit height. For two cubits [in height], it is 144 cubits and 4 ninths. The Sea was 5 cubits high, the first 3 being square and the top two being circular. So the volume of the lower square three cubits being 3 hundred [cubic] cubits, the volume of the entire Sea would be 444 [cubic] cubits and 4 ninths. . . . When you multiply by 4 and a half [*bat*s per cubic cubits], it comes to two thousand, in accord with the scriptures: "its capacity was 2000 *bat*s."[45]

. . .

The [religious authorities] figured a *miqve* as one part of 150 of the Sea. Since the . . . *bat* is 3 *seʾa*s, the Sea was 6000 *seʾa*s. Since a *miqve* is 40 *seʾa*s, it is one part of 150 of the Sea. The volume of 3 [cubic] cubits is 13 *bat*s and a half, which are 40 *seʾa*s and a half.[46] According to this calculation, the size of a *miqve* of 40 *seʾa*s is a cubit by a cubit at a height of 3 cubits less one part in 27 of a cubit.[47] If [the religious authorities] added to the size of a *miqve* [saying it is one by one by three cubits], they did so for strictness, as was their manner.

[41] Kings I, 7:26.

[42] The precise value is $9\frac{3}{5} \times 3\frac{1}{7} = 30\frac{6}{35}$.

[43] This calculation is based on 4 fingers per palm and 6 palms per cubit (as opposed to the 5 palms per cubit above—both possibilities occur in the scriptures). Then, given $\pi = 3\frac{1}{7}$ and the above outer circumference, to obtain an inner circumference of 30 cubits, we require the rim's thickness to be $\frac{6}{35} \times \frac{1}{2\times 3\frac{1}{7}}$ cubits. Multiplied by 24 fingers per cubit, this makes $\frac{36}{55}$ fingers, which is approximately $\frac{2}{3}$ of a finger.

[44] The actual product is $\frac{30}{2} \times \frac{9\frac{3}{5} - 2\times\frac{3}{110}}{2} = 72 - \frac{9}{22}$ (where $\frac{3}{110}$ cubits are $\frac{36}{55}$ of a finger, according to the previous calculation). This error is carried over from Bar Ḥiyya.

[45] Kings 1, 7:26.

[46] According to the above, $\frac{2000}{444\frac{4}{9}} = 4\frac{1}{2}$ is the number of *bat*s in a cubic cubit, so a *miqve* which consists of 3 cubic cubits has $3 \times 4\frac{1}{2} = 13\frac{1}{2}$ *bat*s. A *bat* is 3 *seʾa*s, so a *miqve* is $3 \times 13\frac{1}{2} = 40\frac{1}{2}$ *seʾa*s.

[47] $1 \times 1 \times 3$ cubits give 40 *seʾa*s and a half, so $1 \times 1 \times \left(3 - \frac{1}{27}\right)$ cubits would give precisely 40 *seʾa*s.

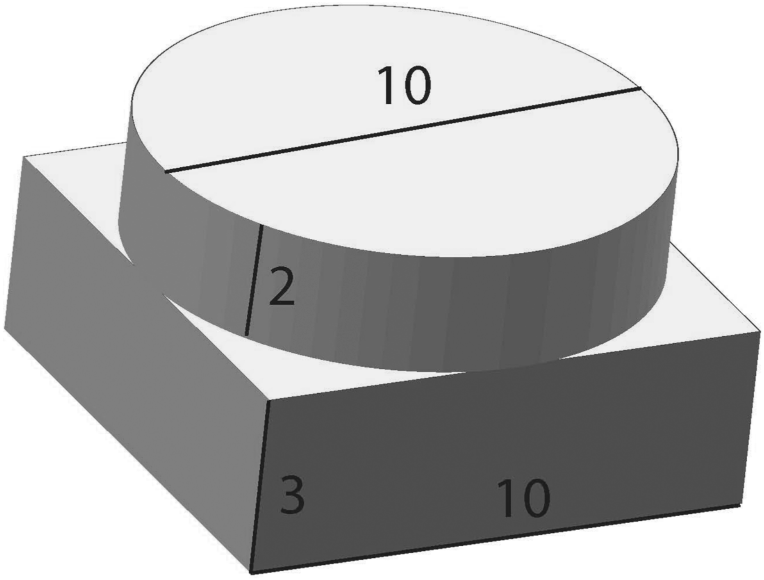


Fig. III-4-1. Solomon's Sea according to the Talmud.

Do not think that the Talmudic authorities had figured the square-shaped three cubits [bottom part of the Sea] to be 300[48] [cubic] cubits, and 150 for the circular two cubits [top part of the Sea], with the diameter of the circle ten whole cubits, because then they would have been wrong in two ways, which is not the case, God forbid. First, if the diameter had been ten whole cubits, the volume . . . [of the circular part would have been 2 x 5^2 x 3 1/7 $=$] 157 and a seventh. But they figured for the two cubit [high] circular part only 150 cubits [Fig. III-4-1].

Second, if this had been the case, then they would have neglected the missing palm in the diameter, whereas it is explicitly written that it was one palm thick. How could they say something so far from the truth as this? That is impossible. But, it is my opinion that they had a doubt whether the missing palm was on all sides (the diameter being 10 cubits less two palms), or in the diameter (leaving 10 cubits less a palm). The volume per one cubit [height] of the Sea, according to the scholars of geometry [that is, $\pi = 3\frac{1}{7}$], with a diameter of 10 cubits less a palm (based on six palms per cubit, which they agreed upon for strictness),[49] would be 76 [cubic] cubits approximately; the volume per one cubit [height] of the Sea with two palms removed from the diameter, would be 73 [cubic] cubits.[50] They took an average, and per one cubit [height] of the Sea, they figured 75 [cubic] cubits; per two cubits [height], they figured 150 [cubic] cubits.

[48] The text reads 450, but this is clearly an error.

[49] Recall that a cubit sometimes had 5 palms and sometimes 6.

[50] $\frac{3\frac{1}{7}}{4} \times \left(10 - \frac{1}{6}\right)^2 \approx 76$; $\frac{3\frac{1}{7}}{4} \times \left(10 - \frac{2}{6}\right)^2 \approx 73$.

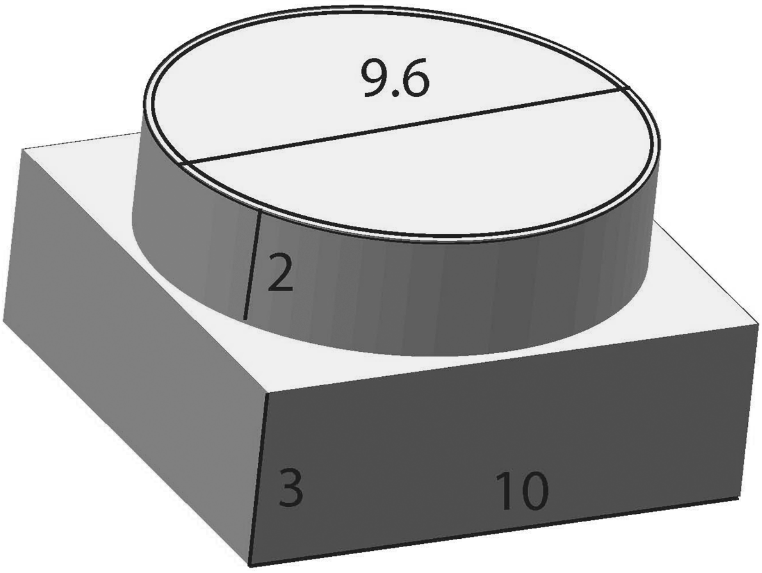


Fig. III-4-2. Solomon's Sea according to Bar Ḥiyya and Enbellshom.

We proceed with several excerpts from Ben Ṣemaḥ's response (leaving out his critique of the calculation error and the wavering between 5 palms and 6 palms per cubit). First, concerning the revised estimate for a *miqve*:

I do not see this as correct. Your error is that you assume that *ḥazal*[51] extracted the volume of a *miqve* from Solomon's Sea, and this is not so. The scriptures do not state that the Sea measured 150 *miqve*s. *Ḥazal* first figured that the volume of a *miqve* is cubit by cubit and 3 cubits high exactly, with no approximation. Indeed, they learned from "he shall bathe his body in water"[52] that a person's entire body should be submerged in a *miqve*, and they estimated the size of an average man to be one cubit by one cubit, three cubits tall.

. . .

Then they needed to figure Solomon's Sea (which contained 2000 *bat*s, which are 150 *miqve*s) in terms of cubits, given the rate of one cubit by one cubit, 3 cubits high [per *miqve*]. To match this, they stated that only the top two cubits were round, and the 3 bottom cubits were square. If it were all round, it would not have held 2000 *bat*s of liquid.

Second, concerning the attribution of 10 cubits to the width of the square part, rather than the diameter of the circular part:

Your interpretation, that the ten cubits from rim to rim are not the inner measure, but that of the three square lower cubits, as arises from your words, contradicts the scriptures.

[51] Hebrew acronym for "the Scholars, Blessed be their Memory." This refers to the religious authorities from the time of the second temple to the closing of the Talmud.

[52] Leviticus 16:24.

Indeed, the scriptures say[53] that from rim to rim in the circular part the width was ten cubits. The scriptures do not mention the square part—this is the interpretation of *ḥazal*, in order to fit, as I mentioned, the two thousand *bat*s content. . . . Since the scriptures do not mention the squareness, how can you interpret that the [10 cubits] width concerns that which is not mentioned?

. . .

It seems that you think that the scriptures gave two magnitudes, one for width and one for circumference. And as you see that these two magnitudes do not fit according to the geometers [that is, with $\pi = 3\frac{1}{7}$], and would contradict each other if they concerned the same object, so you interpret them as concerning two [different] objects. You assigned the width to the square part, and the circumference to the circular part, and make up what is missing in the scriptures, namely that the circumference of the square part is 40 cubits and the width of the circular part is 9 cubits and three fifths approximately. This is fitting to your intention, and would have been acceptable by the mind, if the words of *ḥazal* had not forced us to reject it. You turned things around to serve your purpose, as I will show you well in accordance with the scriptures.

Ḥazal, whose words you seek to correct, would not be pleased by them at all, and would reject them. They explicitly said that the width of ten cubits relates to the inside of the circular part, as was stated in *Eruvin* [14a]. . . . Thus your building upon this interpretation is "like a spreading breach that occurs in a lofty wall, whose crash comes sudden and swift."[54]

Third, the attempt to reckon with the thickness of the rim is rejected.

I said that you turned things round to serve your intention. This relates to your attempt at forcing from the scriptures that the width of the circular part was 9 cubits and 3 fifths approximately, saying that the circle was a palm away from the edge of the square, which makes two palms in diameter, counting five palms per cubit; the two palms, you say, are to be removed. But I cannot find this statement in *ḥazal*, and you too wrote that they were in doubt as to whether the missing palm was in all directions, or referred to the entire diameter, making half a palm on each side. . . . Your vision that the circle is a palm away from the edge of the square is prophetic, because the opinion of *ḥazal* and the literal text of the scriptures is that the width, which is a palm, is so from bottom to top, and does not subtract anything neither at the top or the bottom, since the measurement [of the diameter] is taken on the inside. But close to the rim, this palm dwindles so that at the rim it is [as thin as] a flower. The excess thickness of one palm is on the outside.

Your interpretation also suffers from this, that the ten cubits width in the square part, you measure from the inside, discarding the thickness [of the rim]; but in the circular part you count them with the thickness. . . . Moreover, you add to the thickness two palms as well as the rim, which is like a flower, which contradicts the scriptures.

[53] Kings I, 7:23.
[54] Isaiah 30:13.

Finally, here are Ben Ṣemaḥ's alternative explanations for the discrepancy between the Talmudic calculations and the more precise value of π.

We should say, like the *tosafot*, that this is an approximation, and that it neglects the extra seventh [in the value of π], and that the sages also neglected the extra 7 [cubic] cubits and a seventh [that would arise from replacing $\pi = 3$ by $\pi = 3\frac{1}{7}$ in calculating the volume of a cylinder with diameter 10 and height 2, as above], which is only one part in 21 of the circular part, making approximately two *miqve*s and a third.

This is foreign neither to the way people speak, nor to the scriptures, which we find speaking in mundane manner with imprecise statements. . . . Moreover, it is not foreign to *ḥazal* to make things clearer for their disciples, especially as they follow here the literal meaning of the scriptures and rely on them.

If you wish to follow another way, and retain [the measurements stipulated by *ḥazal* as well as the revised value of π] without any difficulties arising from Solomon's Sea, you can say that the cubits measuring the height were smaller and the cubits measuring a *miqve* larger, as is demonstrated in *Eruvin* [3b]. Between the larger and smaller cubits, you can fit the extra 7 [cubic] cubits and a seventh. . . .

I swear that everything you write about the Sea is very precise. I do not dispute it because it is imprecise, but because I see that it does not reflect the view of *ḥazal*.

5. LEVI BEN GERSHON, *ASTRONOMY*

Wars of the Lord was Levi ben Gershon's (see section I-6) major work on religious philosophy, probably completed by 1330. The *Astronomy*, which survives in separate manuscript copies from the rest of the treatise, forms Book V, part 1 of that work. It was translated into Latin in 1342 by Petrus of Alexandria. Meanwhile, chapters 4 through 11 of the *Astronomy* appeared separately as the treatise *On Sines, Chords and Arcs, and the Instrument "the Revealer of Secrets,"* where the latter refers to what is known as the Jacob Staff.

Calculation of Sines and "heuristic reasoning"

In section 3 of chapter four, Levi constructs his table of Sines in a way similar to Ptolemy's construction of his table of chords in the *Almagest*. That is, he first computes the Sines and chords of many angles using basic geometry as well as the sum formula, the difference formula, and the half-angle formula. For example, the chord of 36° is calculated from Euclid's result on the side of a decagon; then he can calculate the Sine of 18°. From the latter and the well-known Sine of 30°, Levi can calculate the Sine of 12° and therefore the Sines of 6°, 3°, $1\frac{1}{2}^\circ$, and $3/4^\circ$, as well as the Sines of 15°, $7\frac{1}{2}^\circ$, and $3\frac{3}{4}^\circ$.

He next shows how he will compute the Sine of $1/4^\circ$, his desired minimum interval in his table, by a method not strictly "geometrical." This method, "heuristic reasoning" (*heqesh taḥbuli*), is a general one that he uses elsewhere in the *Astronomy*. Levi much preferred a "demonstrative" argument, based on Euclidean precepts, but realized that in certain cases this was impossible. So one alternative was a type of "conditional reasoning," based on making certain assumptions, then checking their consequences, and then repeating until one reached the solution as closely as desired by successive approximation [Mancha, 1998]. For this particular calculation, Levi explains that the tables of Sines with intervals of 1° have errors of as much as 15 minutes of arc when trying to determine an arc corresponding to a given Sine, especially for arcs near 90^o. With his method, he believes that "no perceptible error arises from linear interpolation."

We can also easily find the Sine of $1/4°$ by heuristic reasoning [*heqesh taḥbuli*]. For once the Sine of $8\frac{1}{4}^{\circ}$ [from the Sine of $(15+1\frac{1}{2}^{\circ})$] is known, we also know the Sine of $4\frac{1}{8}^{\circ}$, and, therefore, by successive halvings, we will know the Sine of $\left(\frac{1}{4}+\frac{1}{128}\right)^{\circ}$. Similarly, from the Sine of (4–$1/4°$) [= $3\frac{3}{4}^{\circ}$] we can proceed until we find the Sine of $\left(\frac{1}{4}-\frac{1}{64}\right)^{\circ}$. When we investigated this in this way, we found that the ratio of the Sine of $\left(\frac{1}{4}+\frac{1}{128}\right)^{\circ}$ to the Sine of $\left(\frac{1}{4}-\frac{1}{64}\right)^{\circ}$ is very nearly equal to the ratio of the first arc to the second one, in such a way that there is no difference between these ratios even to the fourth sexagesimal place, although they differ slightly in the fifth. Therefore, we established [by heuristic reasoning, that is, linear interpolation] that the Sine of $1/4$ is 0;15,42,28,32,7. From this amount we can find all the remaining Sines.

In chapter 49, Levi further explains the method of heuristic reasoning. Although the method as described here is the classical method of "double false position,"[55] Levi's main contribution is that one can iterate this method in nonlinear contexts to approximate the desired result. In other words, after calculating a value by means of double false positioning, one can use it as input for a further iteration of double false positioning, and so on. (See [Plofker, 2002] for more details on such iterative approximation in India and Islam.)

It is appropriate to know that it is not possible to do in a quick way a demonstrative research in order to show how our model must be constructed. . . . Therefore, the investigations which lead us to the truth necessarily are of the kind of heuristic reasoning, which are made from trial and investigation, which approach step by step to the truth until it is reached. These types of reasoning belong to the category of conditional reasoning, and there are two classes of them, one of which is taken from an excess and a defect; the second one is taken from two investigations in excess or from two investigations in defect.

For illustrating the first class, we say: if when we considered a determined first quantity, it followed an equation greater than that we have by a given second quantity, and if when we supposed a certain third quantity, it followed from it an equation smaller than that we have by a given fourth quantity, it is known, according to the [rules of] proportion, that it is necessary to suppose a mean [quantity] between the first and the third, so that the ratio of the difference between the first and the mean to the difference between the first and the third is equal to the ratio of the second to the sum of the second and the fourth.[56]

To illustrate the second class, we say: if when we supposed a certain first quantity there followed from it an equation greater than that we have by a given second quantity, and if when we supposed a certain third quantity there followed from it an equation greater than that we have by a given fourth quantity, which is smaller than the second one, it is known, according to the proportion, that it is necessary to suppose a fifth quantity so that the third is the mean between the first and the fifth, and the ratio of the difference between the first and the fifth to the difference between the first and the third is equal to the ratio of the second to the difference between the second and the fourth. And one proceeds in

[55] See section V-3 below and section II-1 in Chapter 3 for more on the double false position.

[56] Suppose you wish to find a such that $f(a) = b$. Suppose further that $f(a_1) - b = b_1$ and $b - f(a_2) = b_2$. Then assuming that f is affine, $\frac{a_1 - a}{a_1 - a_2} = \frac{b_1}{b_1 + b_2}$.

a similar way if the second and fourth quantities are derived from equations smaller than those we have.[57]

The conditional reasoning would be simple when we say: if when we suppose a certain first quantity, there followed from it an equation of a given second quantity not equal to the third quantity which we have, it is known, according to the proportion, that it is necessary to suppose a fourth quantity in such a way that the ratio of the fourth to the first one is equal to the ratio of the third to the second.[58]

At this point the Latin manuscript has a long marginal note, not included in the Hebrew original, which clarifies the previous passage with some examples.

Example of excess and defect: the first [quantity] is 10, that will produce 30, which is 6 [units] greater [than what we have], and this is the second; the third is 6, that will produce 15, which is 9 [units] smaller [than what we have], and this is the fourth; the fifth is 8;24, that will produce 24, what we have.[59]

Example of diverse excess: the first [quantity] is 10, that will produce 36, which is 12 [units] greater [than what we have], and this is the second; the third is 8, that will produce 30, which is 6 [units] greater [than what we have], and this is the fourth; the fifth is 6, that will produce 24, what we have.[60]

Example of diverse defects: the first [quantity] is 6, that will produce 24, which is 12 [units] smaller [than what we have], and this is the second; the third is 8, that will produce 30, which is 6 [units] smaller [than what we have], and this is the fourth; the fifth is 10, that will produce 36, [which is] what we have.[61]

The previous cases are [examples of] composite conditional reasoning; the following one is simple. The first [quantity] is 10, which produces 30, which is the second, which is not equal to 24, which is what we have, and it is the third. Therefore, 8, which is the fourth and whose ratio to 10 is equal to the ratio of 24 to 30, produces 24.[62]

Solving triangles by means of sine tables

In section 5 of chapter four, Levi treats the solution of triangles. He begins with the procedures for right triangles and then moves on to general plane triangles. Certainly, his methods were not new, as they were available in various Islamic trigonometries. Nevertheless, this was one of the earliest treatments in Europe of the basic methods for solving plane triangles. The methods were then applied later in his work for solving astronomical problems.

The procedure for finding the angles and sides of a triangle when some of them are known.

If two sides of a right triangle are known, the remaining side and angles may be found. Let *ABG* be a right triangle, two sides of which are known; I say that the remaining side and

[57] Suppose you wish to find a such that $f(a) = b$. Suppose further that $f(a_1) - b = b_1$, $f(a_2) - b = b_2$, and $b_2 < b_1$. Then assuming that f is affine, $\frac{a_1 - a}{a_1 - a_2} = \frac{b_1}{b_1 - b_2}$. Similarly for $f(a_1), f(a_2) < b$.

[58] This is the simpler case of the Rule of Three, which applied to a linear f.

[59] You wish to find a such that $f(a) = 24$. Suppose that $f(10) - 24 = 6$ and $24 - f(6) = 9$. Then a is 8;24 (sexagesimal) (or 8 2/5).

[60] You wish to find a such that $f(a) = 24$. Suppose that $f(10) - 24 = 12$ and $f(8) - 24 = 6$. Then a is 6.

[61] You wish to find a such that $f(a) = 36$. Suppose that $36 - f(6) = 12$ and $36 - f(8) = 6$. Then a is 10.

[62] This relates to the case of a linear f. If $f(10) = 30$, and we wish to get the result 24, the argument of the function should be $10 \times \frac{24}{30} = 8$.

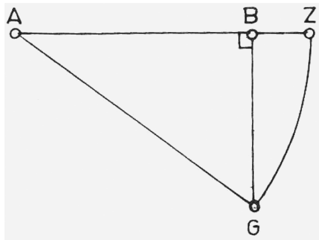


Fig. III-5-1.

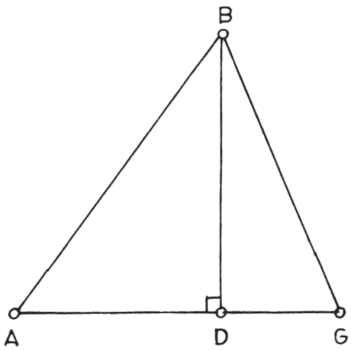


Fig. III-5-2.

angles are also known. Whichever two sides are known, the third side is known because the square of the side that is the hypotenuse is equal to the sum of the squares of the two remaining sides. Thus, if they are known, it is known; and if it and one of the remaining sides are known, the third side is also known because its square is the difference between the square of the hypotenuse and the square of the other side.

Let us assume that angle *ABG* is a right angle, and that sides *AG* and *BG* are known [Fig. III-5-1]. I say that angle *BAG* is known, for if we consider point *A* as center, draw an arc *GZ* with *AG* as radius, and join line *ABZ*, it follows from the above that line *BG* is the Sine of arc *GZ*. Since lines *AG* and *BG* are known, it follows that line *BG* is known in the measure where line *AG* is the semidiameter of 60. In this measure the arc corresponding to line *BG* considered as a Sine can be looked up in the table of arcs and Sines. When arc *GZ* is found in this way, angle *BAG* is also known, as is clear from Euclid. From it angle *BGA* is also known, because it is the complement in 90°, inasmuch as angles *BAG* and *BGA* together are 90°.

If all sides of any triangle whatever are known, its angles are also known. Let the sides of triangle *ABG* be known; I say that its angles are also known.

We drop perpendicular *BD* from point *B* to line *AG* extended if necessary—in the first figure [Fig. III-5-2] point *D* falls within the triangle, and in the second figure [Fig. III-5-3] it falls outside the triangle. I say that the amount of *GD* is known. When we take the excess of the squares of lines *GB* and *GA* over the square of line *AB* in the first figure, or the excess of the square of line *AB* over the squares of lines *GB* and *GA* in the second figure, and divide it by twice line *GA*, the result is equal to line *GD*—as will be clear with a little thought concerning Book II of Euclid [*Elements* II.12, II.13]—and thus the amount of line

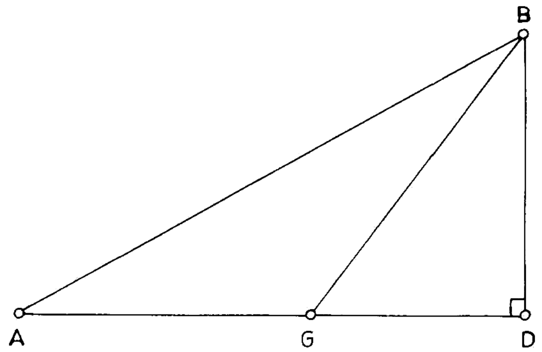


Fig. III-5-3.

GD is known. Since the square of line *GB*, which is known, is greater than the square of line *BD* by [the amount of the square of line *GD*], the amount of line *BD* is known. Therefore, it follows as before that all the angles of right triangle *BDG* are known. In the first figure this yields angle *BGA* and one part of angle *GBA*, namely angle *GBD*, and in the second figure angle *BGA* is known because angle *BGD*, its supplement in two right angles, is known; also angle *GBD* is known. Moreover, since lines *AG* and *GD* are known, the amount of line *AD* is known. Thus all the sides of right triangle *BDA* are known, and therefore angle *BAG* is known in both figures. The remaining angle *GBA* in the triangle is known because angle *GBD* is known and angle *DBA* is known, from which it follows with a little thought that angle *GBA* is known in both figures. Therefore it is clear that all sides and angles of triangle *ABG* are known, and this is what we sought to demonstrate.

If we know two sides of any triangle whatever and one angle such that one of the known sides subtends it, the other angles and the third side are known.[63]

Let the two known sides be lines *AB* and *BG* in triangle *ABG*, and let angle *BAG* be known [Fig. III-5-4]. I say that line *AG* is known and that the remaining angles are known. Let us circumscribe circle *BAG* about triangle *BAG*, and let us consider the diameter of the circle to be line *AD*. Since angle *BAG* is known and we consider it as an inscribed angle, . . . arc *BG* is known. Therefore, the amount of the chord of this arc may be found from the table of arcs and Sines in the measure where line *AD* is 120, and so the ratio of line *BG* to line *AD* is known. Since the ratio of line *BG* to line *AB* is also known, the ratio of line *AB* to line *AD* is known. It follows that line *AB* is known in the measure where line *AD* is 120, and thus arc *AB* may be found in the table of arcs and chords. Since both arcs *BG* and *AB* are known, the remaining arc *GA* is known, from which angle *BGA* and line *GA* are known by the aforementioned procedures. Thus it is clear that the sides and angles of triangle *ABG* are all known, and this is what we sought to demonstrate.

You ought to understand from this explanation that if you consider the diameter of the circle to be 60 and the circumference to be 180°, it is not necessary to compute the chords of these arcs in a table of arcs and chords, because the table of arcs and Sines serves this purpose inasmuch as the Sine is half the chord of twice the arc. The ratio of

[63] Although Levi is discussing what is now known as the "ambiguous" case, he is assuming that in any particular problem, one of the unknown angles is assumed to be acute or obtuse, so there is only a single solution.

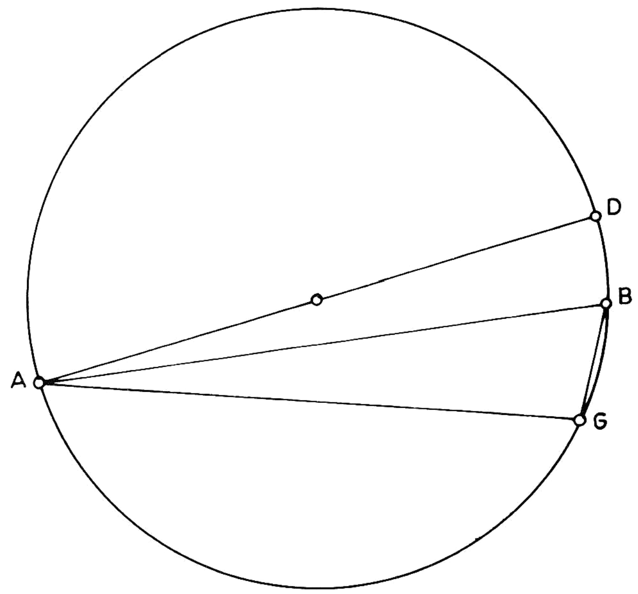


Fig. III-5-4.

the Sine of any arc to the semi-diameter of the circle is equal to the ratio of the chord of twice that arc to the diameter of the circle. Therefore in this proof we chose the second alternative, so that our remarks would be brief. We mention this here to avoid confusion in our subsequent proofs. From this theorem it follows that in any triangle whose sides are straight lines, the ratio of one side to another is equal to the ratio of the Sines of the angles that they subtend. It also follows with a little thought that if the angles of a triangle with straight sides are known and one side is also known, the remaining sides are known because their ratios to the known side are known.

If two sides of any triangle are known, and the included angle is also known, the remaining angles and sides are known.

Consider triangle *ABG* whose sides *AB* and *BG* are known and angle *ABG* is also known; I say that line *AG* is known and that the remaining angles are known. If angle *ABG* is a right angle, this is clear from the preceding. Moreover, if it is acute as in the first figure [Fig. III-5-5] or obtuse as in the second figure [Fig. III-5-6], line *AG* is known. Let us draw perpendicular *AD* from point A to line *BG*, extended if necessary. In either case it is clear that angle *ABD* is known because either angle *ABG* or its supplement in two right angles is known. There remains angle *DAB* which is known because it is the complement in a right angle. Therefore all angles and one side of triangle *ABD* are known, and the rest may be found. Moreover, lines *AD* and *DG* are known in both figures. The amount of line *AG* in right triangle *ADG* is known, and thus the angles of triangle *ADG* are known including angle *AGD*. It was already assumed that angle *ABG* is known; there remains angle *BAG* which is known because it is the supplement in two right angles, and this is what we sought to demonstrate.

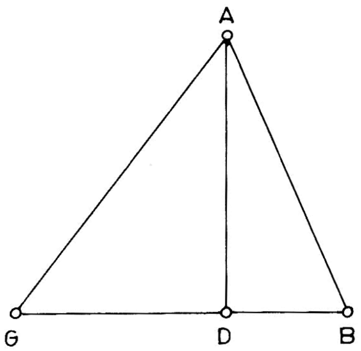


Fig. III-5-5.

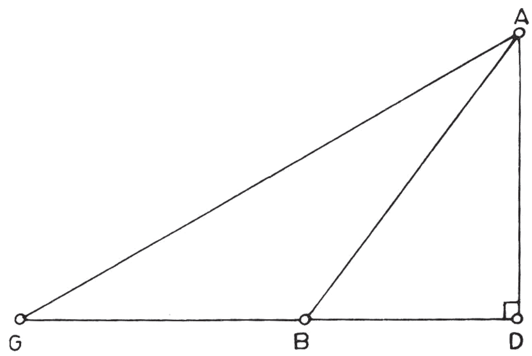


Fig. III-5-6.

IV. SCHOLARLY GEOMETRY

This section is dedicated to geometry in the tradition of Euclid, Apollonius, and Archimedes. We start from Levi ben Gershon's attempt to reduce the parallel postulate to a simpler, more intuitive axiom. We continue with Qalonymos ben Qalonymos's discussion of regular polyhedra, Bonfils's circle measurement, and one of the discussions of the asymptote of the hyperbola. To conclude, we bring the work of one of the most enigmatic and unique Hebrew mathematical authors: Abner of Burgos (also known as Alfonso di Valladolid), who, in his treatise on squaring the circle, which survives only in a fragment, made interesting contributions to the quadrature of lunes and the Western reemergence of the conchoid.

1. LEVI BEN GERSHON, *COMMENTARY ON EUCLID'S ELEMENTS*

Levi ben Gershon's (see section I-6) commentary on Books I–V of Euclid's *Elements* was probably written shortly after 1337. We do not know Levi's sources, but there were numerous commentaries on the *Elements* written in Arabic starting in the tenth century, some translated into Hebrew, as well as commentaries written in Latin. The one to which Levi's work seems closest is *The Book Explaining the Elements of Euclid*, originally attributed to

Naṣīr al-Dīn al-Ṭūsī (1201–1274), but more recently attributed to his son or one of his students [see Rosenfeld, 1988, pp. 80–85; and Lévy, 1992, pp. 90–91]. Levi's commentary deals with selected definitions, postulates, and theorems from the first five books of the *Elements*.

Here we only present the commentary on the parallel postulate (postulate 5), with Levi's proof of the postulate beginning with his own more "self-evident" postulates. Levi's first postulate is essentially that a straight line can be extended to make it greater than any given straight line. His second postulate is that if the two lines in question (in Euclid's postulate 5) form an acute and a right angle, respectively, with the cutting line, then they approach one another on the side of the acute angle and grow farther apart in the opposite direction. (The latter part is omitted from the formulation but is used in the proof.) This leaves two tasks: (1) to show that when the two intersection angles summing to less than two right angles are obtuse and acute, the lines still approach one another (this is achieved in lemma 9), and (2) that the two approaching straight lines actually intersect (shown in the final proof). (See section IV-5 of Chapter 3 for a "proof" of the parallel postulate by al-Maghribī.)

Euclid says: if a straight line falls on two straight lines, forming [on one side] two interior angles less than two right angles, then, if the lines are prolonged on the same side, they will intersect.

Levi says: This proposition is very profound; it is not easy to validate it. In fact, it is not widely accepted that if the two interior angles are less than two right angles, one being obtuse and the other acute, then the two lines intersect. By the same token, the following statement may not be considered as one of the common notions: if a [straight] line falls on two straight lines, forming on one of the two sides two interior angles less than two right angles, then it follows that any other straight line falling on those [straight lines] forms also on the same side two interior angles less than two right angles. Rather, since in the man of subtle intelligence who immerses himself deeply in this science, this produces a doubt, the more so for any novice, the understanding of this premise is not easy. Furthermore, it has been demonstrated in this science [geometry] that [it is possible] for two lines having between them, at the beginning, a certain distance, that they approach each other as they are extended but never meet, even when prolonged to infinity.[1] This premise can also be subject to doubt, although we acknowledge that it is generally recognized that such lines approach one another.

The premise in question is particularly necessary to this science, as is seen with proposition 29 of the first book of this work [Euclid's *Elements*] and the following—from that premise one derives the properties of parallel lines and the equality of the sum of the three angles of any triangle with two right angles. Thus, if it were to fail, the geometry in its totality, or in its major part, would fail. This is why we thought it appropriate to establish it by a proof. This demonstration takes place after the preceding twenty-eight propositions of the book, since Euclid did not have recourse to this premise for any of these propositions.

We pose as a preamble two well-known premises. The first is that which Euclid has mentioned in the fifth book: in essence, he states that it is possible to multiply any line to obtain a line greater than a certain given line. This premise is very clear; even more so given all that had been delineated before.

[1] See section IV-5 on the hyperbola and its asymptote.

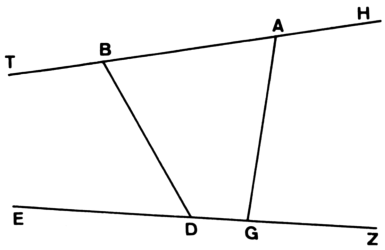


Fig. IV-1-1.

The second premise: the straight line which is inclined [to another straight line] approaches [the second line] on the side where an acute angle is formed [with a line crossing both of these that is a perpendicular from the first line to the second].[2] This is established by considering the sense of the definition; in fact, the notion of inclination means nothing other than the fact that [one line] approaches [the other] in the direction in which they are inclined.

It follows that two straight lines drawn to form two acute angles [with the same straight line] approach one another in the direction [of the acute angles], since each of these is inclined to the other, the notion of acute angle expressing the fact that the line [forming one of the sides of the acute angle] is inclined in the direction [of the line forming one side of the other acute angle]. It follows that on the opposite side they move away, for they approach one another on this side; and also, because on that second side, they depart from the two obtuse angles and each is inclined to the side opposite to that of the other. This is clear and there can be no further doubt as to its truth.

Here ends the commentary on the beginning of [Euclid's] book.

. . .

Here are the propositions that are necessary to demonstrate that: if a straight line falls on two straight lines and forms, on one of the sides, two interior angles less than two right angles, then the lines, if they are prolonged on the same side, will intersect. These propositions have their place after the first 28 propositions of the first book [of Euclid's *Elements*].

[Lemma 1:] There does not exist any quadrilateral figure having all its angles obtuse or having all its angles acute.

Let the quadrilateral be *ABGD*; we claim that it is impossible that all angles be acute or all obtuse. Proof of this impossibility: Suppose that it is possible and assume first that the angles are acute. Prolong the line *AB* in a straight line in two directions to the points *H* and *T*; and also prolong the line *GD* in a straight line in two directions to the points *Z* and *E* [Fig. IV-1-1]. Since each of the two angles *TAG* and *AGE* are acute and thus the angles

[2] This addition to Levi's words helps make sense of Levi's assumption [Lévy, 1992, p. 101]. Tony Lévy justifies it by noting that in Levi's *Treatise on Geometry*, he defines the "approach" of one line to another in terms of a line cutting both. In addition, this addition seems necessary to allow Levi to reach the conclusion in the next paragraph.

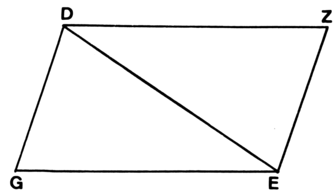


Fig. IV-1-2.

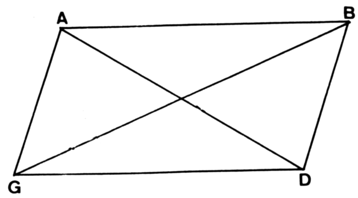


Fig. IV-1-3.

HAG and *AGZ* are obtuse, the two lines *ZE*, *HT* separate from each other in the direction of the two points *H* and *Z* and approach each other in the direction of the two points *T* and *E*. Also, since each of the two angles *ABD* and *BDG* are acute, each of the two angles *TBD* and *BDE* are obtuse, and thus the two lines *HT*, *ZE* separate in the direction of the points *T* and *E*, and approach each other in the direction of the points *H* and *Z*. They therefore separate on the side of the two points *E* and *T*, but also approach on the side of the two points *E* and *T*. This contradiction is impossible. Thus the four angles of the figure *ABGD* cannot all be acute. In the same way, one demonstrates that it is impossible that each of the four angles of the figure *ABGD* is obtuse. As a consequence, there cannot be a quadrilateral figure of which all the angles are obtuse or of which all the angles are acute. QED

[Lemma 2:] We wish to construct a quadrilateral figure for which the opposite sides, taken in pairs, are equal to each other.

Given two straight lines *DG*, *GE* of arbitrary length, meeting at the angle *DGE*, we draw the straight line *DE* [Fig. IV-1-2]. Construct on the line *DE* the triangle *EZD*, where each line is congruent to the corresponding line of triangle *DGE*, that is, the line *EZ* to the line *GD*, and the line *ZD* to the line *GE*. The figure *GDEZ* is thus a quadrilateral of which the opposite sides, taken in pairs, are equal to each other. We have therefore constructed a figure of which the opposite sides, taken in pairs, are equal to each other. QED

[Lemma 3:] Every quadrilateral figure in which the opposite sides are equal to one another also has the opposite angles equal.

Let the quadrilateral *ABGD* have the side *AB* equal to the side *GD* and the side *AG* equal to the side *BD*; I say that the two opposite angles *GAB*, *GDB* are equal and the opposite angles *ABD*, *AGD* are also equal [Fig. IV-1-3]. The proof: Draw the two straight lines *AD*, *BG*. Since the two sides *AB*, *BD* are equal to the two sides *GD*, *AG*, each to its corresponding side [in the triangles *ABD*, *DGA*], and the base *AD* is common, thus the angle *AGD* is equal to the angle *ABD*, to which it is opposite. Similarly, since the lines *AG*, *AB* are equal to the lines *BD*, *DG*, respectively, and the base *BG* is common, thus the

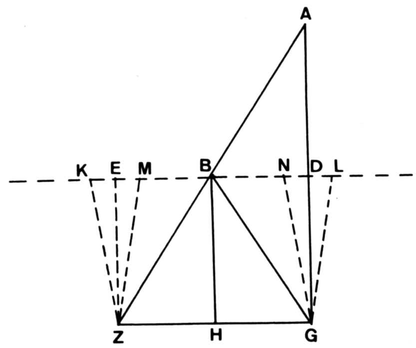


Fig. IV-1-4.

angle *GAB* is equal to the angle *GDB*, to which it is opposite. It is thus established that the quadrilateral figure *ABGD* has opposite angles equal. Thus, every quadrilateral figure of which the opposite sides are equal in pairs has the opposite pairs of angles equal. QED

[Lemma 4:] Given an isosceles triangle, if one extends one of the two equal sides in a straight line from their point of intersection by a distance equal to the original length of the side, and if one draws the base [of the triangle thus formed], this latter forms a right angle with the original base.

For example, let the isosceles triangle be *ABG*, the lines *AB*, *BG* being equal to each other. The straight line *AB* is extended up to *Z* such that *BZ* is equal to one of the two lines *AB*, *BG*; the line *GZ* is drawn [Fig. IV-1-4]. I say that then the angle AGZ is right.

The proof: Divide the line *AG* into two halves at the point *D*, and draw the line *DB*. It results from the eleventh proposition of the first book [of Euclid] that the angle *ADB* is right, and the same for the angle *GDB*. Extend the straight line *DB* to *E*, such that the line *BE* is equal to the line *DB*, and draw the line *EZ*. Since the two lines *EB*, *BZ* are equal, respectively, to the two lines *DB*, *BA*, that is, the line *DB* to the line *BE*, and the line *AB* to the line *BZ*, and since the two vertical angles [at *B*], *DBA*, *EBZ* are equal, then the base *EZ* [of triangle *EBZ*] is equal to the base *AD* [of triangle *ABD*], and the remaining angles [of the first triangle] are equal to the remaining angles [of the second triangle], each to its corresponding one. As a consequence, the angle *BEZ* is equal to the angle *ADB*, and the angle *ADB* is right, so the angle *BEZ* is right. Since the line *EZ* is equal to the line *DA*, which is equal to the line *GD*, [then] the line *EZ* is equal to the line *GD*.

I say that the line *GZ* is [also] equal to the line *DE*. The proof is that it cannot be otherwise; if it were otherwise, then the line *GZ* would either be greater than the line *DE* or would be less than this line. Suppose, first, that the line *GZ* is greater—if this were possible—and divide the line *GZ* into two halves at the point *H*. Since the line *GZ* is greater than the line *DE*, the line *GH*, half of the line *GZ*, is greater than the line *DB*, half of the line *DE*. Also, in the same way, it becomes clear that the line *HZ* is greater than the line *BE*. Extend *BD* in a straight line to *L* so that *BL* is equal to *GH*, and extend *BE* in a straight line to *K* so that the line *BK* is equal to the line *HZ*. Since the two lines *GH*, *HZ* are equal to each other, the two lines *LB*, *BK* are equal to each other. Also, since the line *LB* is equal to the line *GH* and the line *BK* is equal to the line *HZ*, therefore the entire line *LK* is equal to the entire line *GZ*. And since the line *BL* is equal to the line *BK*, and

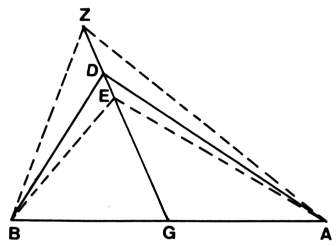


Fig. IV-1-5.

the line *BD* is equal to the line *BE*, it follows that the line *DL* is equal to the line *EK*. The two lines *LD*, *DG* are thus equal to the lines *KE*, *EZ*, each to its corresponding one, that is, the line *LD* to the line *KE*, and the line *DG* to the line *EZ*; and the right angle *LDG* is equal to the right angle *KEZ*; as a consequence, the base *GL* [of triangle *GLD*] is equal to the base *ZK* [of triangle *ZKE*]. Therefore, the figure *LGZK* is a quadrilateral of which the opposite sides are equal in pairs; therefore the opposite angles are equal, the angle *KLG* being equal to the angle *KZG*, and the angle *LKZ* being equal to the angle *LGZ*. As the angle *GDB*, exterior to the triangle *GLD*, is right, the interior angle *GLD* [of the triangle] is less than a right angle; in the same manner, it is shown that the angle *ZKE* is less than a right angle. Since the two angles *GLK*, *LKZ* are acute, the two opposite angles [in the quadrilateral] are [also] acute. Thus, the quadrilateral *GLKZ* has all its angles acute, and this is false. As a consequence, the line *GZ* is not greater than the line *DE*.

The proof that line *GZ* cannot be less than line *DE* is similar.

It has already been demonstrated that [*GZ*] is not greater than [*DE*]. As a consequence, the line *GZ* is equal to the line *DE*. But the line *GD* is also equal to the line *ZE*. The figure *DEZG* is therefore a quadrilateral of which the opposite sides, taken in pairs, are equal; the opposite angles are therefore [also] equal. But the angle *DEZ* is right, and therefore the angle *DGZ* is right. QED

[Lemma 5:] In every right triangle, if the side opposite the right angle is divided into two halves and a straight line is drawn from the point of division to the right angle, then the straight line which results is equal to each of the parts of the divided line.

Let *ABD* be a right triangle with angle *ADB* the right angle. The line *AB* is divided into two halves at the point *G* and the straight line *GD* is drawn. I say that the line *GD* is equal to each of the two lines *AG*, *GB* [Fig. IV-1-5].

The proof is that it cannot be otherwise. We suppose that the line *GD* were either greater than each of the two lines *AG*, *GB* or that it were smaller. To begin, suppose that it is greater, if that were possible. A line is cut off along the line *GD* equal to each of the two lines *AG*, *GB*, say, *GE*, and the two straight lines *AE*, *EB* are drawn. Since the triangle *AGE* is isosceles, if one of the two sides, suppose the line *AG*, is extended from the intersection of the two sides [*GE*, *GA*] an equal length, to give the line *GB*, and the straight line *EB* is drawn, then the angle *AEB* is right [lemma 4]; but the angle *ADB* is [also] right. One has thus constructed on one of the sides of the triangle *ADB*, that is, the line *AB*, two straight lines drawn from the extremities, meeting in the interior of the triangle and subtending an angle equal to the angle subtended by the two other sides [of triangle *ADB*]—the right angle *ADB* being equal to the right angle *AEB*—and this is false

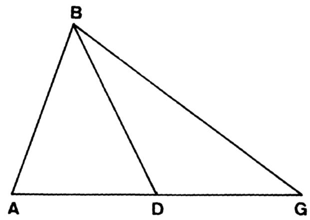


Fig. IV-1-6.

[*Elements* I.21]. By consequence, the line *GD* is not greater than each of the two lines *AG*, *GB*.

The proof that line *GD* is not smaller than the lines *AG*, *GB* is similar.

But it has already been demonstrated that [*GD*] is not greater than each of the lines *AG*, *GB*. So the line *GD* is equal to each of the lines *AG*, *GB*. QED

[Lemma 6:] In every right triangle, the non-right angles are together equal to a right angle.

Let the right triangle be *ABG*, of which the angle *ABG* is right. I say that the two angles *BAG*, *BGA*, taken together, are equal to the angle *ABG*, which is right [Fig. IV-1-6].

The proof: *AG* is divided into two halves at the point *D*, and the straight line *DB* is drawn. Then the line *DB* is equal to each of the two lines *DA*, *DG*; the triangle *ADB* is therefore isosceles. Also, the triangle *BDG* is isosceles. Thus, the angle *DBG* is equal to the angle *DGB*, and the angle *DBA* is equal to the angle *DAB*. Therefore, the two angles *DAB*, *DGB*, taken together, are equal to the angle *ABG*, which is right. QED

[Lemma 7:] In every rectilinear triangle, the three angles [together] are equal to two right angles.

The proof: It cannot but be otherwise that either [the triangle] is a right triangle or it is not a right triangle. If it is a right triangle, the property is proved by virtue of the previous proposition. So assume that it is not a right triangle; I claim that the three angles are [also] equal to two right angles.

Example: Let the non-right triangle be *ABG*. The perpendicular is drawn from the point *A* to the base *BG*, supposed unlimited; let the perpendicular be *AD*. If the perpendicular falls on the line *BG* between the points *B* and *G*, which is the case in the first figure [in Fig. IV-1-7], it is clear that the three angles of the triangle *BAG* are [together] equal to two right angles.

The proof: Since the triangle *ADG* is a right triangle, the two angles *DAG*, *DGA* are together equal to a right angle; it is likewise established that the two angles *DAB*, *DBA* are [together] equal to a right angle. The three angles of the triangle *BAG* together are therefore equal to two right angles.

Similarly, suppose that the perpendicular *AD* falls outside of triangle *ABG*, which is the case in the second figure [in Fig. IV-1-7]. I claim that the three angles of triangle *BGA* are [together] equal to two right angles. The proof: Since the triangle *ADB* is a right triangle, the two angles *DAB*, *DBA* are together equal to a right angle; but the angle *DAB* is itself equal to the [sum of] the two angles *DAG*, *GAB*, and the three angles *DAG*, *GAB*, *DBA* are therefore [together] equal to a right angle. Also, since the triangle *ADG* is a right

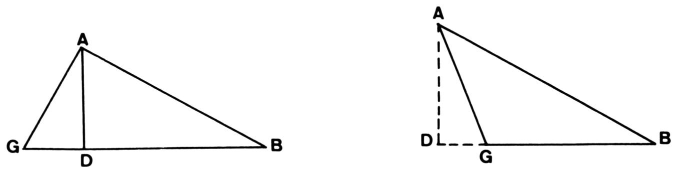


Fig. IV-1-7.

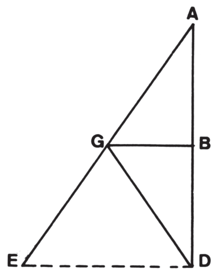


Fig. IV-1-8.

triangle, the two angles *DAG*, *AGD* are equal to a right angle. The three angles *DAG*, *GAB*, *ABG* being [together] equal to the two angles *DAG*, *AGD*, the common angle *DAG* is subtracted; the angles *GAB*, *ABG* which remain are [together] equal to the remaining angle *AGD*. The angle *AGB* is added [to both]. Then the two angles *AGB*, *AGD* [together] are equal to the three angles *AGB*, *GAB*, *ABG* [together]. But the two angles *AGB*, *AGD* [together] are equal to two right angles; the three angles of triangle *ABG* are therefore also equal to two right angles.

It has thus been demonstrated that the three angles of any rectilinear triangle are equal to two right angles, but this conclusion is not required for the result that we seek.

[Lemma 8:] In every right triangle, if one of the sides containing the right angle is extended by a length equal to itself, and if the side opposite the right angle is extended in the same direction a length equal to itself, and if a straight line is drawn connecting the extremities of the lines thus obtained, then this line makes a right angle with the extension of the line containing the right angle.

Let *ABG* be the triangle, in which the angle *ABG* is right. The line *AB* is extended to *D*, so that the line *BD* is equal to the line *AB*. The line *AG* is extended in a straight line to *E*, the line *GE* being equal to the line *AG*, and the straight line *DE* is drawn [Fig. IV-1-8]. I say that the angle *ADE* is right.

The proof: Draw the straight line *GD*. Since the right angle *ABG* is equal to the right angle *GBD*, and the line *AB* is equal to the line *BD*, and the line *BG* is common [to the two triangles *ABG*, *GBD*], then the base *GD* is equal to the base *AG*. The triangle *AGD* is therefore isosceles, and one of its equal sides, the line *AG*, [has been extended] a length equal to itself, that is, *GE*. As a consequence [Lemma 4], the angle *ADE* is right. QED

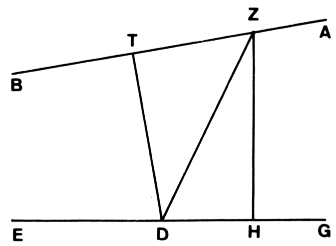


Fig. IV-1-9.

[Lemma 9:] If a straight line falls on two straight lines and forms on one side two interior angles that together are less than two right angles, and if from the vertex of one of the angles a perpendicular is drawn to the second line, then the perpendicular forms, with the other line, an acute angle on the side where the two interior angles are less than two right angles.

Let the two straight lines be supposed unlimited, the lines *AB*, *GE*, and, falling on them, the straight line *ZD* forms [with these] two angles *BZD*, *ZDE*, [together] less than two right angles. I claim that if from point *Z* a perpendicular *ZH* is drawn to the line *GE*, supposed unlimited, the obtained angle *HZB* is acute, that is, the angle that is determined on the side on which the two interior angles are less than two right angles [Fig. IV-1-9]. Similarly, if from point *D* a perpendicular *DT* is drawn to the line *AB*, supposed infinite, the angle *TDE* will be acute.

The proof: Since the three angles of triangle *ZHD* are equal to two right angles, and since the two angles *ZDH*, *ZDE* are also equal to two right angles, the angle *ZDE* is equal to the two angles *ZHD*, *HZD* [taken together]. The angle *BZD* is added [to both]. The two angles *BZD*, *ZDE* are therefore equal to the three angles *ZHD*, *HZD*, *BZD*. But the two angles *ZDE*, *BZD* are [together] less than two right angles; thus the three angles *ZHD*, *HZD*, *BZD* are [together] less than two right angles. But the angle *ZHD* is right. It follows that the two angles *HZD*, *BZD* are [together] less than a right angle, so the angle *HZB* is acute.

Similarly, it is proved that the angle *TDE* is acute. In fact, the angle *BTD*, which is right, is equal to the two angles *TZD*, *TDZ*, since the latter, taken together, form a right angle. The angle *TDE* is added [to both]. The two angles *BTD*, *TDE* are therefore equal to the three angles *TZD*, *ZDT*, *TDE*. But the three angles *TZD*, *ZDT*, *TDE* are less than two right angles. Therefore the two angles *BTD*, *TDE* are less than two right angles. But the angle *BTD* is right; so the remaining angle *TDE* is less than a right angle. QED

[Proof of the postulate of parallels:]

If a straight line falls on two straight lines, forming on one side two interior angles [together] less than two right angles, then the two straight lines, if they are prolonged indefinitely on the same side, will meet.

Let the two straight lines be *AB*, *GD* and, falling on them, the straight line *AE*, forming two angles *BAE*, *AED* [together] less than two right angles. I claim that the two lines *AB*, *GD*, if they are prolonged indefinitely in the direction of *B*, *D*, will meet [Fig. IV-1-10].

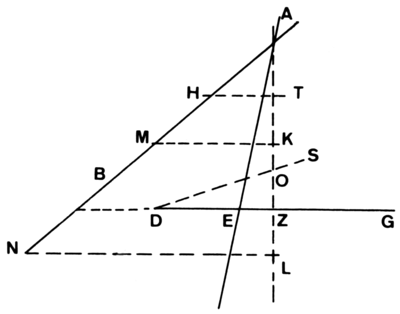


Fig. IV-1-10.

The proof: The perpendicular is constructed from the point *A* to the line *GD*, supposed unlimited; this is the line *AZ*. It is clear from that which precedes that the angle *ZAB* is acute [Lemma 9]. A point, say, the point *H*, is marked anywhere on the straight line *AB*, and from this point the perpendicular to the straight line *AZ* is constructed; this is the line *HT*. Since the angle *ZAB* is acute, it is clear that the point *T* falls on the line *AZ*, extended on the side of *Z*. It has been established that the line *AT*, if multiplied sufficiently many times, will be greater than the line *AZ*. The line *TK* is chosen to be the same length as the line *AT*, and the line *KL* the same length as the line *AK*. When this is done several times, this [line so constructed] will be greater than the line *AZ*.

Suppose that the line *AL* is greater than the line *AZ*. Since the angle *ATH* is right, it is clear that, if the straight line *AH* is prolonged in a straight line an equal length, say, *HM*, and in the same manner the straight line *AT*, giving the line *TK*, then the line that joins the point *K* and the point *M*, that is, the line *KM*, meets the line *AK* in a right angle, as has already been proved [Lemma 8]. Similarly, it is clear that, if the straight line *AM* is prolonged in a straight line an equal length, giving the line *MN*, and the straight line *NL* is drawn, then the angle *ALN* is right.

But since the angle *AZD* is right, the angle *LZD* is also right; the two lines *ZD*, *LN* are therefore parallel on the side [of the triangle *ALN*; *Elements* I.28]. Since the straight line *ZD* is interior to the triangle *ALN*, and the triangle *ALN* is finite, it is therefore possible to prolong the straight line *ZD* continuously in a straight line, such that it exits the triangle *ALN*; but it is not possible that the straight line *ZD*, prolonged indefinitely, exits between the two points *L* and *N*, since the two lines *ZD*, *LN*, which are parallel, would meet, and this is false. I say also that the straight line *ZE* may not exit between the two points *L* and *A*; if this were possible, then it would also be that the line *ZD* would cut the line *AL* at *O*, and in this case the two straight lines *ZO*, *ZDS* would encompass a surface, and this is false. This being so, it remains only that the straight line *ZD*, when it is prolonged indefinitely, must exit the triangle *ALN* between the two points *A* and *N*. As a consequence, the straight line *ZD* meets the line *AN*. QED

2. LEVI BEN GERSHON, *TREATISE ON GEOMETRY*

The *Treatise on Geometry* was probably written shortly after the commentary on the *Elements*. Levi wanted to construct geometry on a stronger foundation than Euclid's.

Unfortunately, the only extant manuscript of this work contains just the first 24 definitions and hypotheses. We present here the twenty-third of these, which distinguishes between the necessary boundedness of actual lines in the world and the indefinite extensibility of mathematical lines.[3]

23. I say that it is possible to prolong a limited straight line in a straight line, continuously, without any determined limit. In fact, the line is always augmentable while still being limited.

This hypothesis has been subjected to doubt. In fact, the Philosopher [Aristotle] contradicts it in his celebrated book *On Physics*. He has explained the impossibility of supposing that a line can be greater than the line that is the greatest that is contained in the universe, that is, the world in its totality. In fact, a line only has existence in a body. Since this premise is very necessary for the geometer, it is appropriate for us to investigate the doubt that befalls it. We say that the aspect through which the Philosopher has contradicted this premise is the necessity for a body to be finite, as has been explained in the place mentioned; but since a line has existence only in a body, it is necessary that the magnitude of the line be limited, that is to say, it is impossible that it can be greater than the straight line contained in the universe, that is, the world in its totality.

This being so, we should investigate from what aspect it is necessary for a body to be finite and limited, that is, not greater than the body of the world. We say that it is manifest that this is a necessity for a body inasmuch as it is a physical body, as has been proven there. However, the geometer supposes that the line is unlimited from the point of view of augmentation, in the sense that one may always add to that which has been added: this does not relate to a line as it is in a physical body, but inasmuch as it is in a mathematical body. From this aspect, the impossibility of the absence of a determined limit in the body has not been demonstrated from it [the body] being always finite. Indeed, the geometer poses this hypothesis from the perspective that it is possible and not from the perspective that it is impossible. As a consequence, in this perspective, no contradiction follows from this premise.

With this premise posited, the geometer recognizes that it is impossible for any magnitude to be infinite, since it exists in actuality and must necessarily be limited. And we claim that this does not contradict the premise that we have mentioned: in fact, what is necessary for a magnitude inasmuch as it has magnitude—and it is this aspect that the geometer considers—is that it not be infinite, but not that it not be greater than the world. Although we acknowledge that the line is always augmentable indefinitely, it does not necessarily follow that the line is infinite in magnitude. In fact, the line, whatever increments it receives, is finite, and that is always so. As for the absence of a limit that we posit with respect to it—it relates to the possibility of augmenting and does not mean that the line is [actually] infinite in magnitude.

[3]The source of the distinction between an unacceptable actual infinity and a permissible indefinite infinity goes back to Aristotle, who is Levi's obvious reference. A possibly related discussion in a mathematical context can be found in Ibn al Haytham's interpretations of Euclid. His concern is the tension between the finite lines that our imagination can contain, and the indefinitely extended lines required by Euclid's postulates [Sude, 1974, 49–55, 88–90].

3. QALONYMOS BEN QALONYMOS, *ON POLYHEDRA*

Qalonymos ben Qalonymos (see section II-5) translated into Hebrew an anonymous Arabic manuscript on polyhedra early in the fourteenth century, a manuscript extending "Book XIV" of Euclid's *Elements*, written by Hypsicles in the second century BCE. Muhyī al-Dīn al-Maghribī used this same manuscript to produce a new Arabic version of the same treatise.

It is not known whether the original from which each borrowed was an Arabic original or a translation from a Greek original. In any case, although the Qalonymos and the al-Maghribī versions are similar, neither is a copy of the other. They have most of their theorems in common, but the ordering and some of the proofs are different. A substantial part of al-Maghribī's version is in section IV-5 in Chapter 3. Here we present two propositions from Qalonymos's version that are not included in that material (although the propositions do occur in the al-Maghribī version). We have chosen to include these in part to demonstrate that Qalonymos believed that there would be a Hebrew-reading audience for these rather advanced geometrical ideas.[4]

[Proposition 18]: We wish to show that the ratio of the [surface] area of the cube to the [surface] area of the icosahedron is as the ratio of the square of the side of the pentagon[5] to [three] and a third times the equilateral triangle whose side equals the root of three times the square on a decagon of the circle that circumscribes the pentagon.

For example, we take AB as the side of the hexagon[6] and divide it in mean and extreme ratio at C. Let D be equal in power [*maḥziq*][7] to AB and AC [i.e., the square on D is equal to the sum of the squares on AB and AC], so D is the side of the pentagon.[8] Let E be equal in power to AB and BC, that is, the root of three times the square of AC, where AC is the side of the decagon.[9] I say that the ratio of the [surface] area of the cube to the [surface] area of the icosahedron equals the ratio of the square of D to three and a third times the equilateral triangle on E. Its proof is: it has already been shown in the preceding proposition that the ratio of the edge of the cube to the edge of the icosahedron is as D is to E, and the ratio of the [square on the] edge of the cube to the square on the edge of the icosahedron is as the square on D to the square on E.[10] Inverting, the ratio of the square [on the edge] of the cube to the square on D is as the ratio of the square on the edge of the icosahedron to the square on E. [For any two lines, the ratio of the square on the first line to twice the triangle on that line is as the ratio of the square on the second line to twice the triangle on that line.] So the square on D is to twice the triangle on D as the square on E is to twice the triangle on E. Therefore, the square [on the edge] of the cube is to the square on D as the square on the edge of the icosahedron is to the square on E. And [the square on the edge of the cube] is to twice the triangle on D, which is twice

[4]For more details on the two manuscripts and on another Hebrew version, see [Langermann, 2014].

[5]The "pentagon" is the pentagon formed by the bases of five triangles of the icosahedron that have a common vertex. Thus the side of the pentagon is an edge of the icosahedron.

[6]The "hexagon" is the hexagon inscribed in the circle that circumscribes the pentagon.

[7]The Hebrew word *maḥziq*, translated here as "equal in power," is a translation of an analogous Arabic word and also of the Greeek *dunamene*.

[8]*Elements*, XIII.9 and XIII.10.

[9]*Elements*, XIII.4.

[10]This is proposition 7 of *Elements* XIV, the work of Hypsicles; see [Montelle, 2014].

the face of the icosahedron, as the square on D, which is the edge of the icosahedron, to twice the triangle on E. The ratio of six times the square of the [edge of the] cube, which is the [surface] area of the cube, to twelve times the face of the icosahedron, which is three fifths of the [surface] area of the icosahedron, is equal to the ratio of the face of the cube to twice the face of the icosahedron, which is the same as the ratio of the square on D to twice the triangle on E. The [ratio of the] surface area of the cube to three-fifths of the surface area of the icosahedron is equal to the ratio [of the square on D] to twice the triangle on E. The ratio of three-fifths the [surface] area of the icosahedron to the surface area of the icosahedron is equal to the ratio of twice the triangle on E to three and one-third the triangle on E. The complete ratio of the surface area of the cube to the surface area of the icosahedron is equal to the ratio of the square on D to three and one-third the equilateral triangle on E.

It is clear from what we have described that the ratio of three-fifths of the surface area of the icosahedron to the surface area of the cube is the same as the ratio of twice the triangle on E to the square on D.

[Proposition 19]: We wish to show that the ratio of the [surface] area of the icosahedron to the [surface] area of the octahedron is equal to the ratio of five times the square on the side of the decagon of the circle to the square on the side of the pentagon.

Let A be the side of the pentagon and B the side of the decagon. I say that the ratio of the [surface] area of the icosahedron to the [surface] area of the octahedron is equal to the ratio of five times the square on B to the square on A. Its proof is, we take C so that its square is three times the square on B. We have already shown that the ratio of three-fifths of the [surface] area of the icosahedron to the [surface] area of the cube is equal to [the ratio of] twice the triangle on C to the square on A. We have already said that the ratio of the [surface] area of the cube to the [surface] area of the octahedron is equal to the ratio of the square on A to twice the triangle on A, for we have already shown in the preceding that the ratio of the [surface] area of the cube to the [surface] area of the octahedron is equal to the ratio of the side of any equilateral triangle to its altitude,[11] which is like the ratio of the square [of its side] to twice its triangle. The ratio of three-fifths of the [surface] area of the icosahedron to the [surface] area of the cube is equal to the ratio of twice the triangle on C to the square on A. The ratio of the surface area of the cube to the [surface] area of the octahedron is equal to the ratio of the square on A to twice the triangle [on A]. In the equality of the ratio, the ratio of three-fifths of the [surface] area of the icosahedron to the [surface] area of the octahedron is equal to the ratio of twice the triangle on C to twice the triangle on A, which in turn is equal to the ratio of the triangle on C to the triangle on A. The ratio of the triangle on C to the triangle on A is equal to [the ratio of the square on C to the square on A and thus to the ratio of three times the square on B to] the square on A. The ratio of three-fifths of the [surface] area of the icosahedron to the [surface] area of the octahedron is as the square on C, which is the same as three times the square on B, to the square on A. The ratio of three-fifths of the [surface] area of the icosahedron to the [surface] area of the entire icosahedron is equal to the ratio of three times the square on B to five times the square on B. In the equality of the ratio, the ratio of

[11] See proposition 12 in the section IV-5 of Chapter 3.

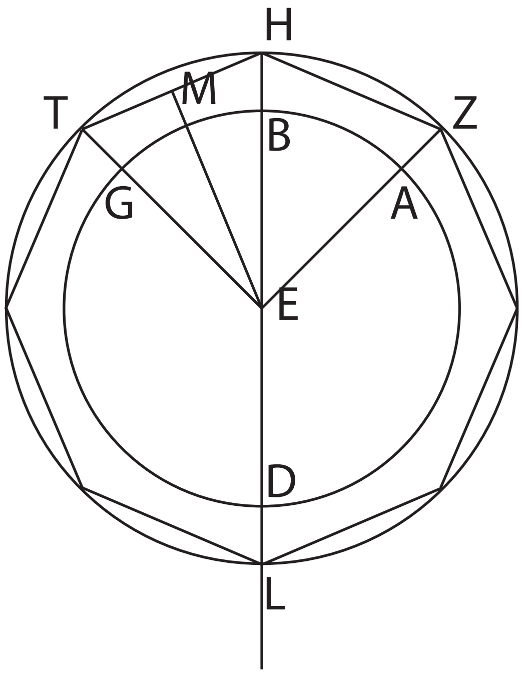


Fig. IV-4-1.

the [surface area of the] icosahedron to the [surface] area of the octahedron is five times the square on B to the square on A.

4. IMMANUEL BEN JACOB BONFILS, *MEASUREMENT OF THE CIRCLE*

This text by Immanuel Bonfils (see section I-3) is a variation on Archimedes's proof for the formula of the area of a circle. It is embedded in a text concerning an evaluation of π (greater than 67,801/21,600 using a 3072-gon), isoperimetric theorems for the circle and sphere, and root extraction. The proof circumvents the construction of a sequence of circumscribed/circumscribing polygons whose areas are arbitrarily close to a given circle; instead it assumes the existence of a circle with a given area and relies on the constructability of a polygon in the ring between any two concentric circles. According to [Lévy, 2012], this proof builds on (and simplifies) the ninth-century proof by *Banū Mūsā*, which relied on constructing a polygon whose circumference is between a given segment and the circumference of a given circle.

Any circle is equal to the product [lit. area] resulting from [the multiplication of] half its diameter by half its circumference. This means that we think of half the circle as if it were a straight line.

Let there be a circle *ABGD* whose center is *E*, and let half its diameter be *EB* and half its circumference arc *BGD*. I say that circle *ABGD* equals the product resulting from the multiplication of line *EB* by arc *BGD* [Fig. IV-4-1].

Proof: If it were not so, let us say that the product resulting from [the multiplication] of line *EB* by arc *BGD* would equal a circle either greater or smaller than the circle *ABGD*.

Let us consider first the greater, that is, circle *ZHTL*, drawn around the center *E*. Inside the circle *ZHTL* let us construct a [regular] polygon with equal angles not touching circle *ABGD*, according to what has been established by [proposition] 13 of [Book] 12 of Euclid [*Elements*], and this figure is *ZHTL*.

Let us draw *EM*, which is the perpendicular to line *HT*. It is evident that the product resulting from [multiplying] *EM* by *MH* is equal to [the area of] triangle *EHT*. This holds for all the triangles constructed in this figure when we draw lines from the center to the vertices of the polygon. Now *MH* is half *HT*. Therefore the product resulting from [multiplying] *EM* by half the perimeter of the polygonal figure equals the entire [area of the] figure *ZHTL*. But *EM* is greater than *EB*, since the [polygonal] figure does not touch the circle [*ABGD*], and since half the perimeter of figure *ZHTL* is greater than half the circumference of circle *ABGD*. Therefore the figure *ZHTL* is greater than the product resulting from [multiplying] *EB* by arc *BGD*, which is half the circumference of circle *ABGD*.

We had posited that [the area of the] circle *ZHTL* equals the product resulting from [multiplying] the line *EB* by arc *BGD*. Therefore the figure *ZHTL* is greater than the circle *ZHTL*. This [however] is false, since the circle *ZHTL* circumscribes the figure *ZHTL* and it exceeds the [polygonal] figure with all the [sectors constituted by the] arcs [subtended] by the sides [of the polygon]. Therefore the product resulting from [multiplying] EB by arc BGD is not equal to a circle greater than circle *ABGD*.

The proof of the other alternative is analogous, and we omit it.

5. SOLOMON BEN ISAAC, *ON THE HYPERBOLA AND ITS ASYMPTOTE*

In his *Guide to the Perplexed* I 73, Maimonides claims that some things that are impossible to imagine are nevertheless true. As an example he gives the hyperbola and its asymptote: these lines approach each other indefinitely but never meet.[12] This argument, which is not original to Maimonides, has a long history in the Latin, Arabic, and Hebrew literature [Freudenthal, 1988].

Maimonides's comment inspired several Hebrew investigations of asymptotes. The text quoted here belongs to a Hebrew transmission, which, according to [Lévy, 1989a,b], stems from the Arabic work of al-Ṭūsī. Its proof that a right angle hyperbola approaches its asymptote is clear and concise, even if this manuscript leaves a small gap: it shows that the hyperbola and the asymptote come "much closer" as they extend, but fails to quantify this relation, and therefore does not formally show that they do come *arbitrarily* close. The means to fill the gap, however, are easily accessible to anyone who can follow the text and formulate the gap in a precise manner.

This text is attributed to Solomon ben Isaac, whose identity has not been traced. The manuscript, which seems to be a sixteenth-century Italian copy written during Solomon's life,

[12] "Know that there are things that a man, if he considers them with his imagination, is unable to represent to himself in any respect, but finds that it is as impossible to imagine them as it is impossible for two contraries to agree. . . . It has been made clear in the second book of the *Conic Sections* that two lines between which there is a certain distance at the outset, may go forth in such a way that the farther they go, this distance diminishes and they come nearer to one another, but without it ever being possible for them to meet even if they are drawn forth to infinity, and even though they come nearer to one another the farther they go. This cannot be imagined and can in no way enter within the net of imagination. Of these two lines, one is straight and the other curved, as has been made clear there in the above-mentioned work" [Maimonides, 1963, vol. I, §73].

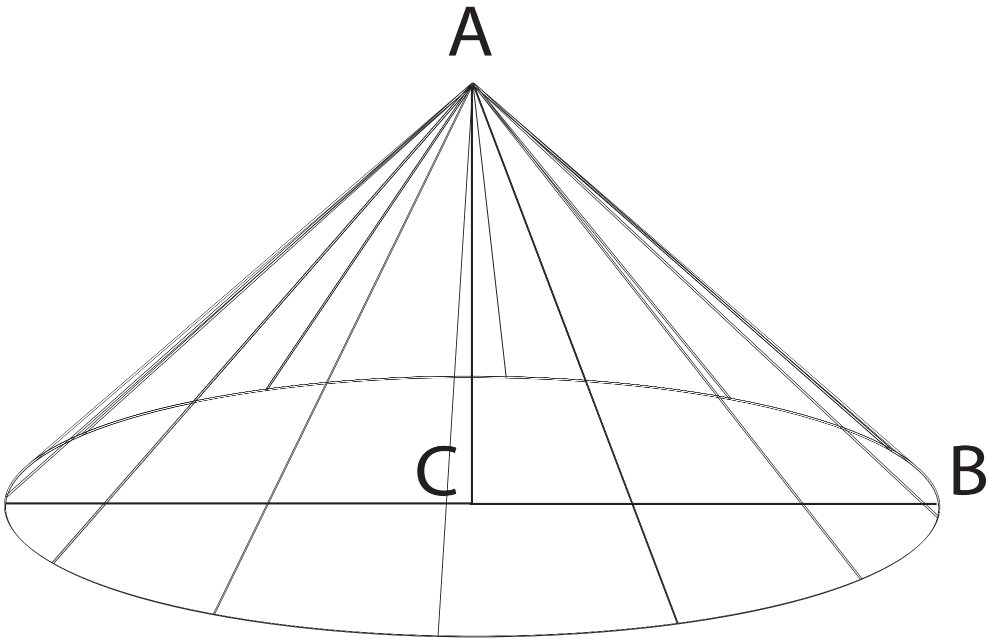


Fig. IV-5-1.

dates his work to around 1500 [Lévy, 1989a]. Among the other treatments of this problem that belong to the same tradition we should mention the text attributed (falsely, according to [Lévy, 1989b]) to Simon Moṭoṭ [Sacerdote, 1893–1894].

In the text here, we omit the preliminary propositions cited from the *Elements*: the angles of a triangle sum to two right angles; an isosceles triangle has equal base angles and vice versa; an isosceles triangle with a right angle has two half-right angles; in a right angle triangle the height is a mean proportional between the parts of the base as cut by the height; if we divide a line segment, the square on the whole segment equals the squares of the parts and twice the rectangle contained by the parts; and the square built on half a line segment with an addition equals the square on half the line segment and the rectangle contained by the addition and the original segment with the addition.

Given a right angle triangle with two equal sides surrounding that angle, if you set one of those sides erect, and turn the triangle around until it returns to its starting position, then the resulting figure is called a cone with circular base. If you cut this figure by a surface parallel to the base, there will be a smaller circle at the intersection.

An example: a triangle *ABC* of that form, with right angle *ACB* and the side *AC* equal to the side *CB* [Fig. IV-5-1]. When you set the side *AC* erect and turn the triangle *ABC* until the point *B* returns to its place, a figure is formed whose base is a circle, and the entire figure is similar to a round cylinder ending at the top at point *A*. This figure is called a cone, which is what we wanted to explain.

For any cone cut by a plane through its axis (which is the height of the initial triangle that you turned), at the intersection there would be a right angle triangle, whose right angle is

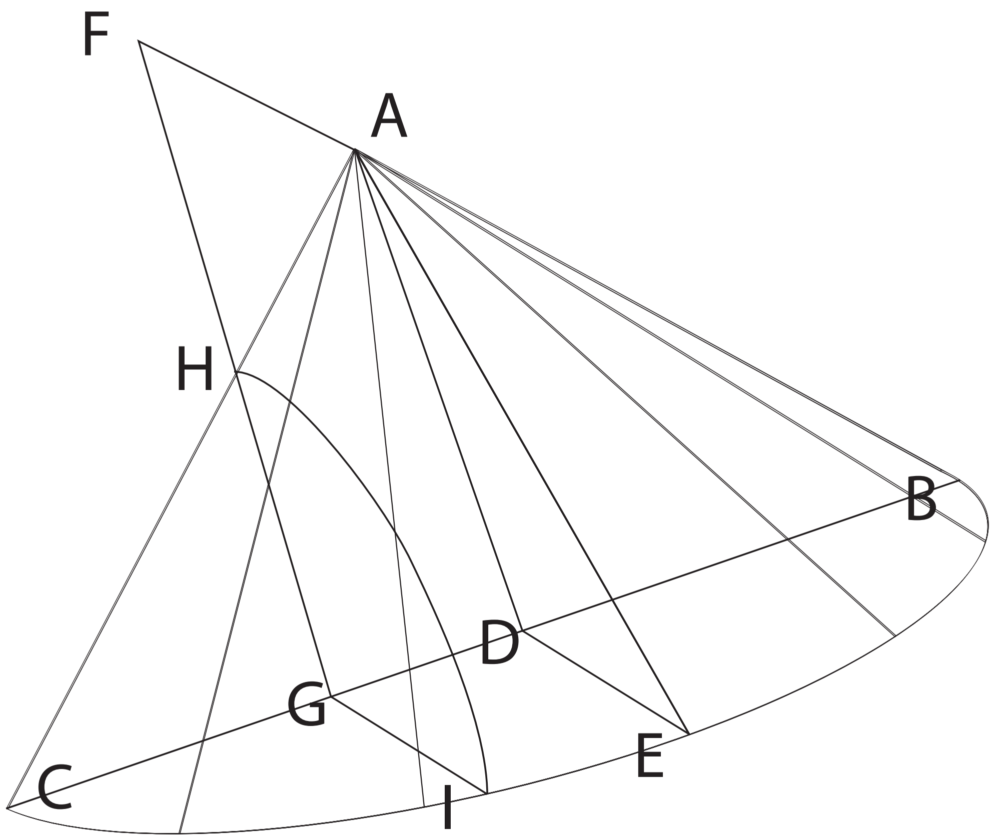


Fig. IV-5-2.

the one at the top of the cone, and at its base there would be half a circle with its diameter. If we set this [solid] figure on the intersection, its base will be the aforementioned triangle, the half circle will rise up high, and the diameter of the half circle will be at the bottom, forming the chord of the right angle of the aforementioned triangle. All this is explained in the figure [Fig. IV-5-2], which is what we wanted to explain.

. . .

If one of the sides of the half cone that surround the right angle is extended, and we draw from one of its points a line parallel to the axis of the figure, going into the triangle (which is the base of the half cone), and touching the diameter of the half circle (which is the chord of the right angle of the base triangle), and from the point of the diameter touched by the aforementioned line you draw a height touching the circumference, and you cut the cone by a plane through the aforementioned height and line, then the plane will form a curved line on the surface of the cone.

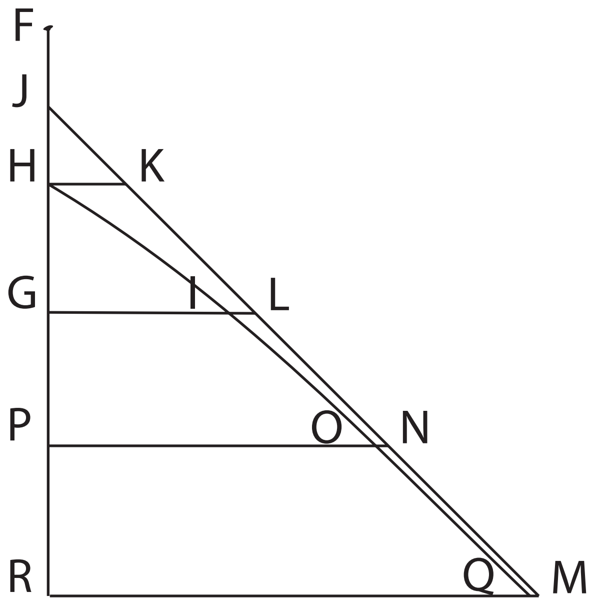


Fig. IV-5-3.

Let the base of the half cone be the triangle marked *ABC*, and the half circle erected on it *BEIC* with its diameter *BDGC* [Fig. IV-5-2]. Having extended the side *BA* out of the triangle to the point *F*, and drawn from the point *F* a straight line *FHG* parallel to the axis *AD*, and drawn from the point *G*, where the line touches the diameter, a height *GI* touching the circumference, then when we cut the half cone by a plane through the height *GI* and the line *GH*, this plane forms a curved line on the surface of the half cone.

To make things clearer, I form a figure consisting of the entire line *FHG* and the height *GI* together with the plane cutting the figure and the following additions [Fig. IV-5-3]. We bisect the line *FH* at point *J*, and set on the point *H* a height *HK* equal to the line *JH*, which is half the bisected line. We draw a line from *J* to *K* and extend it to *M*. We also extend the height *GI* as far as this line at point *L*.

Concerning the former figure [Fig. IV-5-2], we say that the line *BG* on the diameter equals the line *GF* of the triangle *BGF*. Moreover, the line *GC* that remains on the diameter equals the line *HG* of the triangle *HGC*, and the height *GI* is the mean proportional between the two parts of the diameter, *BG* and *GC*.[13] The two lines *BG* and *GC* equal the two lines *GF* and *GH*, respectively. Therefore the square of the height *GI* is equal to the area surrounded by *GF* and *GH*. In the latter figure [Fig. IV-5-3], the line *FH* has already been bisected at point *J* and added to the line *GH*. Therefore the square of *JG* equals the area surrounded by *GF* and *GH* and the square of *JH*.[14]. Now the height line *GL* equals the line *JG*. Therefore the square of *GL* equals the area surrounded by the

[13]This fact was included in Solomon's list of preliminary Euclidean results.
[14]This fact was included in Solomon's list of preliminary Euclidean results.

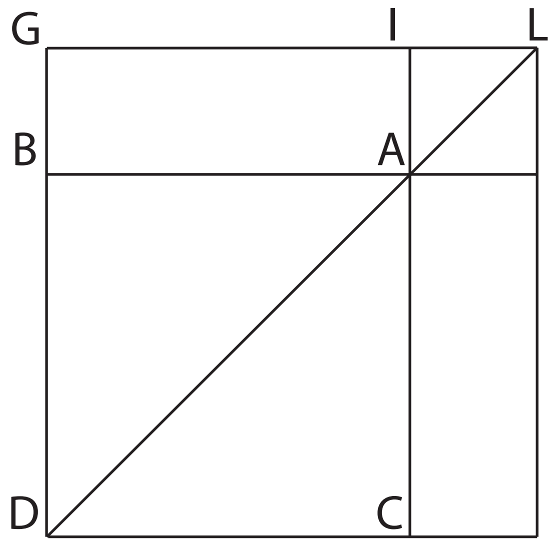


Fig. IV-5-4.

two lines *GF* and *GH* and the square of *JH*. But the square of *GI* has already been said to equal the area surrounded by the two lines *GF* and *GH*. Therefore the square of *GL* exceeds the square of *GI* by the square of *JH*.[15]

Now if you extend the entire diagram so that the line *FJHG* reaches the point *P* and let the height on *P* be *PON*, the former proofs show that the square of *PN* exceeds square of *PO* by the square of *JH*. And if you extend the entire diagram further, so that *FJHGP* reaches *R* and let the height on *R* be *RQM*, the former proofs also show that the square of *RM* exceeds the square of *RQ* by the square of *JH*. And so ever on, even if you extend the diagram indefinitely, one square will exceed the other by the square of *JH*. Therefore they [the hyperbola and asymptote] cannot meet, because if this were possible, then they [the height to the cone and the height to the plane] would be equal, and the square of the one would not exceed the square of the other. But we have explained that the one exceeds the other by the square of JH. Therefore it is false that they ever meet.

The demonstration that as they [the hyperbola and the asymptote] extend they grow nearer, and that their distance diminishes with respect to the initial distance, which is the height *JH*, is explained by what we have already explained in the previous diagrams before the extension, namely, that the square of *GL* exceeds the square of *GI* by the square of *JH*. Let us complete the square to render this visible.

Let the square of *GI* be the square marked *ABCD* [Fig. IV-5-4]. The three complementing areas [which complete the square of *GI* to the square of *GL*, namely, the three rectangles forming the gnomon *ABGILC*] equal the square of *JH*. Thus *JH* is much greater than *IL*. But *HK* has already been said to equal *JH*. Therefore the line *KH* is much larger than the line *IL*. And so the lines [the hyperbola and asymptote] grow much closer.

[15] $GI^2 = GB \cdot GC = GF \cdot GH$; $GL^2 = JG^2 = GF \cdot GH + JH^2$. Therefore, $GL^2 = GI^2 + JH^2$.

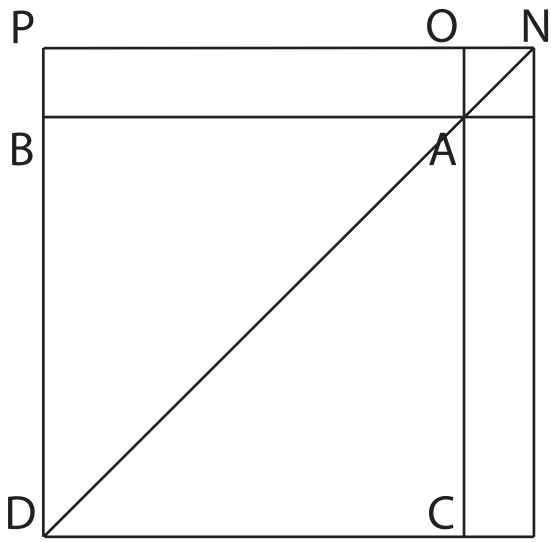


Fig. IV-5-5.

Extending the diagram as far as *P* and the height *PON*, we have already explained that the square of *PN* exceeds the square of *PO* by the square of *JH*. When we complete the diagram [to form a square], let the square of *PO* be the square marked *ABCD* [Fig. IV-5-5]. The three complementing areas with respect to the square of *PN* equal the square of *JH*. Since the square of *PN* is much wider and longer than the square of *GL*, and the three respective complementing areas equal the square of *JH*, therefore the line *NO* is much smaller than the line *IL*, and the lines [the hyperbola and asymptote] grow closer. In the same way it is explained that they grow closer as you extend the entire diagram until it reaches the point *R*, and so on indefinitely.

Therefore two lines are drawn, one of which is the straight line *JLNM* and the other is the curved line *HIOQ*. At their inception they had a certain distance, and as they extend they grow nearer. They cannot ever meet even if they are extended indefinitely, which is what we wanted to explain.

6. ABNER OF BURGOS (ALFONSO DI VALLADOLID), *SEFER MEYASHER ʿAQOV* (*BOOK OF THE RECTIFYING OF THE CURVED*)

This section was prepared by Avinoam Baraness.[16]

Abner of Burgos (1270–1348) was a Jewish scholar from Castile, who converted to Christianity and was known after his conversion as Alfonso di Valladolid. After his conversion, Alfonso became engaged in anti-Jewish activity: polemical writings, public disputations, and

[16] Avinoam Baraness expresses his deepest gratitude to Professor Ruth Glasner for significant encouragement and assistance during the writing process. However, he notes that the section on the quadrature of the lune (prop. 23) is due entirely to Tzvi Langermann, as stated in the list of sources at the end of this chapter.

even inciting the authorities against the Jews.[17] Alfonso was well versed in the Bible and the Talmud, and also in Greek and Arabic philosophy. His philosophical work, *the New Philosophy*, is lost. His extant works are mostly polemical, in Castilian translation. Only one book and a few letters were preserved in the original Hebrew.

The Hebrew mathematical treatise *Sefer Meyasher ʿAqov*[18] by Alfonso is extant in the single manuscript [British Library Add 26984].[19] [Gluskina, 1983] identified the author "Alfonso" as Abner de Burgos—an identification reconfirmed by [Freudenthal, 2005]. *Sefer Meyasher ʿAqov* is very different from the rest of his works known to us. The stated aim of the book is "to inquire whether there possibly exists a rectilinear area equal to a circular area truly, neither by way of approximation as earlier scholars suggested." In the book Alfonso was concerned with the three famous geometrical problems of antiquity, the measurement of curves, plane figures, and solids, and methods for comparing areas circumscribed by mixed straight and curved lines. The treatise contains five chapters, of which the first four prepare the background for the final (and lost) one, where the author was to achieve his goals. The first two chapters offer a historical and philosophical introduction with special attention to the role of motion in geometry. The third chapter consists of 33 geometrical propositions, which are claimed to be "useful for this discipline." Propositions 2–9 of the third chapter are missing, and the text is interrupted at the beginning of the fourth chapter. The original manuscript includes only a few simplistic diagrams for the second chapter.

Whereas Alfonso's philosophical writing sometimes lack clarity and sharpness, the mathematical sections are better organized. The text presented here is from the third chapter, whose content is purely mathematical. Each proposition of this chapter follows the formal Euclidean pattern. It is, however, difficult to point out the organizing principle underlying the chapter as a whole. The propositions deal with a wide range of topics: comparing areas, theorems connecting the length of some segments in polygons (including Ptolemy's theorem on inscribed quadrilaterals), two generalizations of the Pythagorean theorem, characters of compound ratios (in anachronistic terms: products of ratios), two theorems on magnitudes divided into n parts, pre-trigonometric theorems, and some of the classical problems of antiquity. In this selection, we present Alfonso's quadrature of the lune and his doubling of the cube by means of a conchoid.

The quadrature of the lune (proposition 23)

Since the goal of Alfonso's treatise was the quadrature of the circle, one need not be surprised to find there a treatment of the quadrature of the lune. But Alfonso's introduction makes it clear that this quadrature only gives us reason to believe (rather than a proof) that squaring the circle should be possible. We bring here only the second construction that Alfonso

[17] For a detailed documentation of his life and works see [Baer, 1961–1966] and [Sadik, 2012]. Although Abner and Levi ben Gershon were contemporaries, there is no evidence that either was aware of the other.

[18] *The Rectifying of the Curved*, alluding to Isaiah, 40:4 "and the crooked shall be made straight." Clearly, Abner understands the Hebrew in this sense, which is the King James translation, rather than in the sense of the Jewish Publication Society translation.

[19] The manuscript has no colophon and contains many blunders and mistakes. A scientific edition of this manuscript was published by [Gluskina, 1983]. It includes the original text in Hebrew, a Russian translation, and a commentary, along with mathematical remarks by the historian of mathematics B. A. Rosenfeld. Baraness and Glasner are preparing an English translation of the text with mathematical commentary.

derives from Hippocrates, because its treatment is a little more original. The commentary and translation for this proposition are adapted from [Langermann, 1996].

It is particularly interesting that Alfonso knows quite well that Hippocrates is the author of the quadrature of the lune. [Clagett, 1964–1984, III, pp. 1317–1318] has pointed out that "in none of the many manuscripts of the medieval *Quadratura circuli per lunulas* was Hippocrates named as the author." Nor was his name mentioned by Ibn al-Haytham, the only Arabic authority whose writings on the subject have survived, albeit partially [Suter, 1986].

This construction is essentially the same as Hippocrates's second quadrature as reported by Eudemus [Heath, 1981, 192–193]. However, the construction of Alfonso is simplified considerably. Hippocrates's method has in fact two parts, both of which involve constructing trapezia that are then circumscribed by circles, which form the outer circumference of the lune; the second part in particular has drawn much attention from historians, because it contains one of the earliest known *neusis* constructions. In contrast, Alfonso displays but a single step, and, using Ptolemy's theorem (as Gluskina points out in her commentary), proves that his construction is in fact the trapezium described by Hippocrates. Now we have no evidence that this theorem was known before the time of Ptolemy (second century CE; see [Toomer, 1984, pp. 50–51]), so it seems unlikely that it was part of Hippocrates's own procedure, which was then only summarized by Eudemus; nor is it invoked by Simplicius in his discussion.

Proposition 23: There is yet another figure equal to a rectilinear area. We take straight line *AB* such that when multiplied by itself it will be equal to three times line *BC* multiplied by itself.[20] We take line [*AC*] multiplied by itself to be equal to [the sum of] line *BC* multiplied by itself together with the product of *AB* with *CB*. With these three lines we construct triangle *ABC*, as has been explained. On it we circumscribe [circle] *ABCD* with center *E* [Fig. IV-6-1]. Since line *AC* is greater than chord *CB*, arc *ADC* is greater than arc *CB*. From the latter we mark off arc *AD* equal to arc *BC*. We join lines *AD*, *DC*, *EA*, *ED*, *EC*, *EB*, *BD*.

Now the sides of triangle *ABC* are equal to the sides of triangle *ADB*, respectively. Moreover, *AC* and *BD* are equal to each other. Since *ABCD* is a quadrilateral inscribed within a circle, the product of lines *AC* and *BD*, in other words *AC* multiplied by itself, is equal to the product of *AD* and *CB*—which is [the same as] *CB* multiplied by itself—together with the product of *AB* with either *CB* or *CD*. Therefore, *CB* is equal to *CD*.[21] Accordingly, *AB* multiplied by itself is equal to the sum of chords *AD*, *DC*, and *CB*, when each has been multiplied by itself.[22] We construct on *AB* triangle *ABZ* similar to triangle *AE*[*D*]. With [*Z*] as center we draw sector *ZAHB*. It follows that sector *ZAHB* will be equal to sector *EADB*, and segment *AHB* is equal to the [sum of the] three segments *AD*, *D*[*C*], and *CB*. Therefore, lune *ADCBH* is equal to the rectilinear figure *ADCB*.[23] QED

[20] This construction and the one in the next line can be carried out by the procedure given in *Elements* II.14, "To construct a square equal to a given rectilinear figure."

[21] By Ptolemy's theorem, proved earlier in the text, $AC \cdot BD = AD \cdot BC + AB \cdot CD$. Due to the symmetries of *ABCD*, we have $AC^2 = BC^2 + AB \cdot CD$. But *AC* was constructed such that $AC^2 = BC^2 + AB \cdot BC$, so $CD = BC$.

[22] *BC* was constructed such that $AB^2 = 3 \cdot BC^2$. The equality of *AD*, *DC*, and *CB* thus establishes Alfonso's claim. Therefore, similar shapes constructed on *AB* and on *BC* (or *AD* or *DC*) will have an area ratio of 3 : 1, as Alfonso claims.

[23] The lune equals the trapezoid *ADCB* less segment *AHB* plus the sum of the three segments *CB*, *DC*, and *AD*. But these latter three segments equal segment *AHB*, so the lune equals the trapezoid.

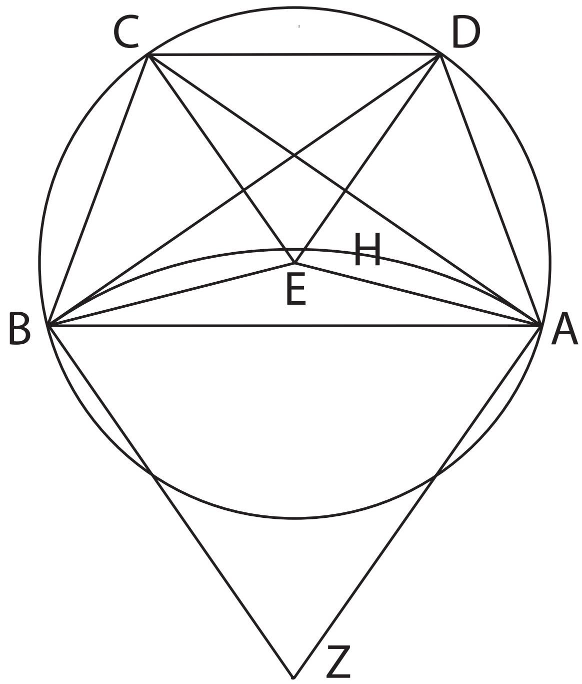


Fig. IV-6-1.

The conchoid of Nicomedes and its applications (propositions 29–32)

Propositions 29–32, which are presented here, seem to form an independent unit, dealing with the conchoid of Nicomedes and some of its uses. Being located toward the end of the third chapter, it can be considered as one of the book's pinnacles. The quoted text is not long but is representative of the third chapter as well as of Alfonso's general mathematical style. This selection is of both historical and mathematical interest: it attests to the fact that the conchoid was known earlier than hitherto assumed, and that Alfonso's methods seem to be relatively complex and somewhat unique to the author.

Given a straight line (the "ruler" or "canon" AB), a point outside it (the "pole" P) and a distance (d), the conchoid of Nicomedes is the locus of all points lying at the given distance d from the ruler AB along the segment that connects them to the pole P.[24]

If P is the origin, and AB is the line $y = a$, then the curve is defined by the polar equation $r = \frac{a}{\sin\theta} + d$. The curve has two branches on opposite sides of the ruler, to which both are asymptotes. The branch passing on the side of the pole has three different distinct forms, depending on the ratio between a and d: If $a < d$, it has a loop (as in Fig. IV-6-2); if $a = d$, then P is a cusp point; and if $a > d$, the curve is smooth. The other branch does not change topologically.

[24] The conchoid can be viewed as the trace of a fixed point on a straight line through the pole, moving along the ruler, but it can also be defined in terms of the *neusis* property as the locus of all the points on any line through the pole whose distance from the ruler along the line is constant. It is not clear whether the latter definition, seemingly not involving motion, can be attributed to Nicomedes himself [Sefrin-Weis, 2010, p. 244, fn. 3]. Yet, as already mentioned, geometric motion was characteristic of Alfonso's approach.

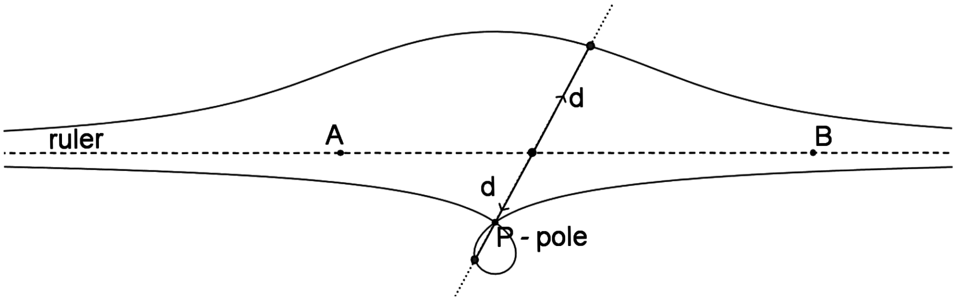


Fig. IV-6-2.

It is generally accepted that the conchoid, whose name means shell form,[25] was invented and first studied by the Greek mathematician Nicomedes (ca. 280–210 BCE).[26] In his treatise *On Conchoid Lines*, known to us from secondary sources,[27] Nicomedes supposedly described the generation of the curve, its classification into types, some of its properties, and a mechanical device for drawing it. Nicomedes also applied the curve to solve two of the classical problems of antiquity: the trisection of an angle and the doubling of the cube (reduced to the problem of finding two mean proportionals).[28] Both solutions depend on a construction that cannot be implemented with compass and ruler but can be implemented with a conchoid.

It is usually taken for granted that all the applications of the conchoid made in antiquity were developed by Nicomedes himself, and that interest in it was revived in the late sixteenth century [Toomer, 2008]. But Alfonso's text presented below provides rare evidence that the conchoid was known and used in the West in the fourteenth century. In proposition 29, Alfonso constructs the conchoid, and in the three following propositions (30–32) he uses it to trisect an angle, find two mean proportionals, and construct a parallelepiped of the same volume as a given parallelepiped that is also similar to another given parallelepiped. The main questions concern the sources of this knowledge and the manner of its transmission. A close study of the text indicates significant differences between Alfonso's approach to the conchoid and the parallels known from Greek sources.

First, Alfonso stated his aim in constructing the conchoid (proposition 29) as the finding of two asymptotic lines, whereas it is accepted that Nicomedes invented the conchoid to trisect an angle and duplicate the cube [Heath, 1981, p. 238; Sefrin-Weis, 2010, pp. 126–128].[29]

[25] The curve was called "conchoid" by Proclus, but Pappus called it "cochloid." Heath claims that the latter was evidently its original name [Heath, 1981, p. 238].

[26] Nothing is known of his life; the dating is estimated by references to his work. See [Heath, 1981, p. 238; Toomer, 2008].

[27] Pappus's *Collection*, Eutocius's *Commentary on Archimedes' Sphere and Cylinder*, and Proclus's *Commentary on Euclid* I. See [Toomer, 2008].

[28] In the case where the greater line is double the smaller line. The reduction was established in the fifth century BCE by Hippocrates of Chios. See [Heath, 1981, pp. 244–246].

[29] But it was also understood that this curve is interesting in itself: Geminus, in his first classification of lines [Heath, 1956. pp. 160–161] includes the conchoid and the hyperbola in the same subdivision, for having the same subtle distinguishing character: they are "asymptotic." Thus it is possible that the asymptoticity of the conchoid had already been adopted as its main character by some scholars before Alfonso.

It should be noted, however, that a simpler example of asymptotes was offered by Apollonius of Perga and was well known in the Jewish world through a remark in Maimonides's *Guide for the Perplexed* I.73 [Freudenthal, 1988; see section IV-5 above). Alfonso probably preferred the more complex example of the conchoid, because he was interested in the uses of this curve as well.

Second, the wording of proposition 29 suggests that *both* branches of the curve are constructed and both of them are used in the following propositions: the external one in proposition 30, and the internal one in proposition 31.

Third, the angle trisection (proposition 30) is achieved in a similar manner to that attributed to Nicomedes [Heath, 1981, p. 235; Sefrin-Weis, 2010, pp. 148–149], but the position of the perpendicular (relative to the angle's side) is different, hence the required trisection is constructed outside the given angle, rather than inside. Of the three applications of the conchoid that Alfonso mentions, only this one resembles one of those known from the Greek tradition.

Fourth, and most interesting, proposition 31 is an impressive construction of two mean proportionals that neither resembles nor alludes to any Greek or Arabic solutions known to us.[30] Moreover, proposition 32 seems to be a unique generalization of the problem of doubling the cube, which does not simply rely on the reduction of the problem to that of finding two mean proportionals[31] but shows how to use these proportionals to construct the doubled cube.

We do not know whether Alfonso acquired his knowledge about the conchoid from an unknown Arabic text based on Pappus's *Collection,* or from some oral tradition. Therefore it may not be determined whether he preserves a tradition unknown to us, or whether propositions 30–32 render his own elaboration, based on a fragmentary acquaintance with the known tradition that goes back to Pappus. Like the question of whether and how he "squared" the circle, this question too remains open.

Proposition 29: We wish to find the origin[32] of two lines, the one straight and the other curved, so that there is a certain [initial] distance between them, but when produced, the distance between their extremities decreases; one of them approaches to the other, but they do not intersect, even if produced indefinitely.

How? We consider two lines AB, BC enclosing a right angle B, and we pick a point D on the line AB either between A and B or beyond B.[33] Then we move point B on the line BC in the direction of point C, so that the line BD is moved with it in such a manner that the point D is opposite to point A, namely, points A, D, B are always collinear [Fig. IV-6-3]. By this motion point D describes a segment of a curved line DGH, which we

[30] See [Clagett 1964–1984, I, pp. 335–345, 658–665, III, pp. 27–30, 849–854, 1163–1179; Heath, 1981, pp. 244–268; Knorr, 1989, pp. 251–319; Rashed, 2011–2014, pp. 60–69, 103–107].

[31] This fact is notably remarkable in light of Heath's comment that since this reduction had been shown, all later mathematicians considered the problem of two mean proportionals rather than the original problem [Heath, 1981, p. 246].

[32] Literally, *yeṣiat shney haqavim.*

[33] The external branch is not mentioned explicitly, but its existence is implied: it is said that point D is picked outside B too. Reading the construction with reference to the letters noted by a prime in the diagram (G' instead of G, etc.) generates this branch.

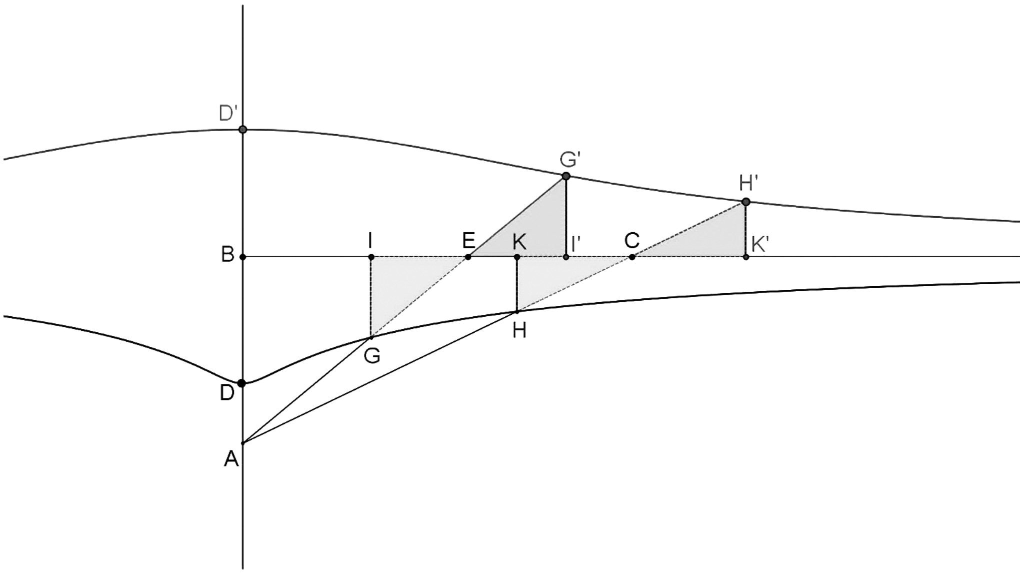


Fig. IV-6-3.

call the conchoid.[34] I say that as long as lines *BC*, *DGH* are produced in the direction of *CH*, the distance between their extremities decreases, and they never meet.

The demonstration: We draw the two straight lines *AHC*, *AGE*. The three lines *BD*, *GE*, *HC* are equal to one another. We draw from the points *G*, *H* two perpendiculars *GI*, *HK* onto the line *BC*. Since (i) the square of the line *GE* is greater than the square of *GI*, (ii) the two triangles *GIE*, *KHC* are right-angled, (iii) the hypotenuses *GE*, *HC* are equal, and (iv) the angle *GEI* is greater than the angle *KCH*, it follows that the perpendicular *GI* is greater than perpendicular *HK*. Similarly it can be shown that of the perpendiculars drawn from the conchoid *DGH* to the straight line *BC*, the closer [perpendicular] to the line *AB* is greater than that more distant from it. Hence, as long as the two lines *BC*, *DGH* are produced in the direction of *CH*, the distance between their extremities decreases and they never meet. QED

[34]The original term is: *haqav haparuṣ*. We are not aware of the use of this word in Hebrew mathematical texts before Alfonso. The root *prṣ* means (1) to expand, to spread out without limits (like the Arabic root *frš*); (2) to crack, to break through (the Arabic root *frḍ* has the related meaning of "to notch," "to make incisions"). In his first classification of lines (given by Proclus; see [Friedlein, 1873, pp. 111; Heath, 1956, pp. 160–162]), Geminus distinguishes between composite and noncomposite lines, and divides the latter class into: (a) those forming a figure (e.g., circle, ellipse, cissoid) and (b) not forming a figure or indeterminate and extending without limit (e.g., straight line, parabola, hyperbola, conchoid). In a second version of the classification [Friedlein, 1873, pp. 176–177; Heath, 1956, p. 160], a fine distinction of the last subdivision is made: "of the lines which extend without limit, some do not form a figure at all, but some *first come together and form a figure, and for the rest, extend without limit*." Following Tannery, Heath concludes almost inevitably that the figure formed is a loop, and that "the curve which has a loop and then proceeds to infinity is a variety of the conchoid itself"—namely, the branch having the loop. These two characteristics of the conchoid may be thought of as associated with the two meanings of *paruṣ* mentioned above (forming a figure—cracked; extending without limit—spreading out).

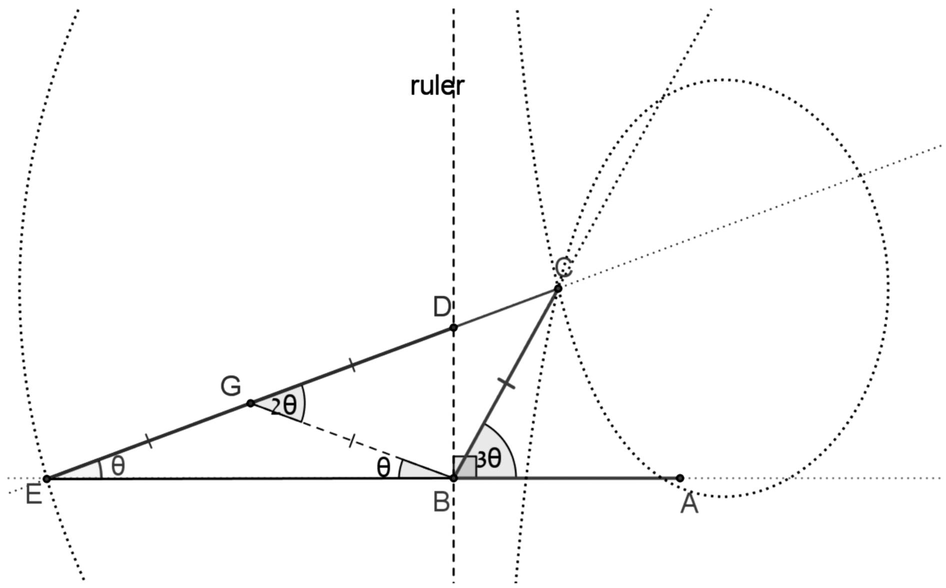


Fig. IV-6-4.

The instrument designed to draw the conchoid is common among craftsmen, and is very useful in this discipline.

Proposition 30: To divide any rectilinear angle into three equal parts.

How? We consider a rectilinear angle *ABC* and erect a perpendicular *BD* upon *AB*, we set the size of *BC* as we wish, and we produce *AB* indefinitely in the direction of *B* [Fig. IV-6-4]. We put the plotting device at the point *C* in such a manner that it meets the two lines *AB*, *BD* at the two points *D*, *E*, such that the line *DE* is twice the line *BC*. This can be done by drawing the conchoid as mentioned above.[35] Then angle *DEB* would be a third of the given angle *ABC*.

The demonstration: Since the line *DE* of triangle *BDE* is the hypotenuse of the right angle *DBE*, and when we cut it at the [mid]point *G* and join *BG*, [the line] *EG* becomes equal to *GB*, which is equal to *DG*,[36] then the angle *BGC*, which is equal to the angle *BCG*, is equal to angle *GBE* together with angle *GEB*. So the angle *DEB* is a third of the two angles *GEB*, *GCB* together, which are equal to the given angle ABC.[37] QED

Proposition 31: We wish to find two straight line segments [that are] mean proportionals between two other straight line segments which are unequal to one another.

How? We let *AB* be the smaller line and *CE* the larger. We describe a semi-circle whose diameter is *CE*, the larger [segment], and whose center is *B*. *D*, the midpoint of *AB*, falls

[35]To trisect angle *ABC*, Alfonso finds a point *E* on the line *AB* (beyond *B*) so that $ED = 2BC$. This is done by means of a conchoid with pole *C*, ruler *BD* and distance 2*BC*. Here Alfonso implicitly assumes that the angle is acute.

[36]Since *BG* is the median to the hypotenuse *ED* of the right-angled triangle *EBD*.

[37]If we set angle $E = \theta$, then angle *BGD* equals 2θ, being an exterior angle to triangle *GEB*. According to the construction, $BC = BG$, so triangle *BGC* is also an isosceles triangle, and the angle *BCD* is 2θ. Therefore the angle *ABC*, being an exterior angle to triangle *BCE*, equals 3θ, and the angle *E*equals one third of the angle *ABC*.

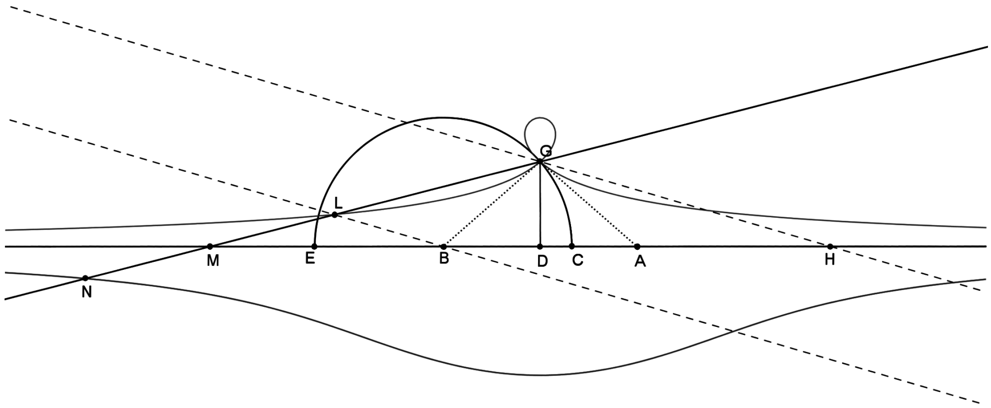


Fig. IV-6-5.

on the diameter [*CE*]. We draw a perpendicular *DG* on the diameter [and extend it] to the circumference of the circle, and we produce *AD* in the direction of *H* until *AH* equals *AB*. We join *HG*, and draw from the point *B* a line *BL* parallel to *HG* and produce it indefinitely. We draw from point *G* a line *GM* toward the line *CEM*, meeting the line *BL* at the point *L*, in such a manner that the line *LM* is half the diameter [Fig. IV-6-5]. This can be done by drawing the conchoid.[38] I say that the ratio of *AB* to *GL* is as the ratio of *GL* to *BM*, and as the ratio of *BM* to *CE*.

Its demonstration: We produce *GM* until *LN* is equal to the diameter. Since the excess of the product of *MG* by itself over the product of *BG* by itself is as the product of *MB* by itself and by twice *BD*,[39] which is also as the product of *GL* by itself and by twice *LM*,[40] then the product of *AM* by *MB* is as the product of *NG* by *GL*.[41] Moreover, the ratio of *AM* to *NG* is as the ratio of *GL* to *MB*, and the ratio of *AB* to *GL* is as the ratio of half *MB* to *LM*, which is equal to the ratio of *MB* to *LN*.[42] It was already established that the ratio of *AB* to *GL* is as the ratio of *AM* to *NG*, which is equal to the ratio of *LG* to *MB*.[43] Hence, the ratio of *AB* to *GL* is as the ratio of *GL* to *MB* and to the ratio of *MB* to *LN*, which is equal to the diameter.

[38] Here we use the conchoid with pole G, ruler CE and distance $BE = \frac{1}{2}CE$. By means of it one may find a line through G, cutting the parallel through B at point L and line CE at point M, so that $LM = BE$. L is actually the intersection point of the conchoid with the parallel through B.

[39] $MG^2 - BG^2 = (MD^2 + DG^2) - (BD^2 + DG^2) = (MB + BD)^2 - BD^2 = MB^2 + 2MB \cdot BD = MB(MB + 2BD)$.

[40] $MG^2 - BG^2 = GM^2 - LM^2 = (GL + LM)^2 - LM^2 = GL^2 + 2GL \cdot LM = GL\,(GL + 2LM)$.

[41] Since $MB + 2BD = MA$ and $GL + 2LM = GN$, the last two footnotes yield $MB \cdot MA = GL \cdot GN$, and consequently $GL : MB = MA : GN$.

[42] HB and GL are segments of rays caught between the parallels HG and BL. By triangle proportion theory, their ratio is the same as that of the segments between the parallel and the origin, $MB : ML$. If we divide both ratios in half, we get $AB : GL = \frac{BM}{2} : LM = BM : LN$.

[43] We already have $AB : GL = BM : LN$, which by proportion theory is the same as $(AB + BM) : (GL + LN) = AM : GN$, which is already known to be the same as $LG : MB$.

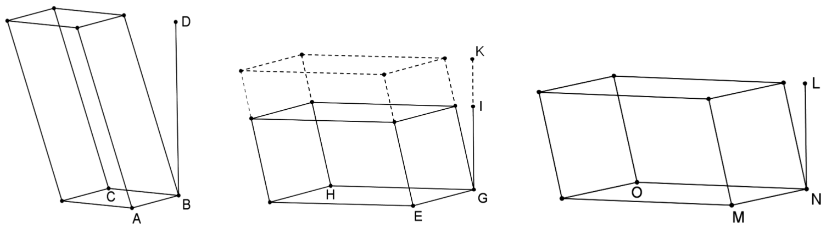


Fig. IV-6-6.

Proposition 32: We wish to construct a polyhedron[44] which is equal [in volume] to a given polyhedron, and which is similar to a second given polyhedron.[45]

How? We set a polyhedron whose base is the area *AB* and whose height is *DB*, and a second polyhedron whose base is the area *EH* and whose height is *IG* [Fig. IV-6-6]. We wish to find a third polyhedron which is equal to the first and similar to the second. We erect upon area *EH* another polyhedron whose height is *GK*, and which is equal to the first polyhedron.[46] We draw between *GK* and *IG* two mean proportionals, *P*, *LN*, so that the ratio of *GK* to *P* is as the ratio of *P* to *LN* and as the ratio of *LN* to *IG*.[47]

We set the ratio of *NM*, which is unknown, to *EG*, which is known, to be as the ratio of *LN*, which is known, to *IG*, which is known. We erect upon the line *NM* an area *MO* which is similar to the area *EH*. It [the area *MO*] would be the base of the required third polyhedron and *LN* would be its height.

Its demonstration: Since the ratio of *GK* to *LN* is as the duplicate ratio of *LN* to *IG*, which equals the duplicate ratio of *NM* to *EG*,[48] which equals the ratio of area *MO* to area *EH*, then the product of *GK* by area *EH* is as the product of *LN* by area *MO*, and the two polyhedra are equal.[49] Therefore the third polyhedron is equal to the first. And since the ratio of *LN* to *IG* is as the ratio of *NM* to *EG*, the third polyhedron is similar to the second.

V. ALGEBRA

The Hebrew literature does not contain much algebra [Lévy, 2003, 2007]. The only explicitly algebraic treatises known are an anonymous algebra in the tradition of al-Khwārizmī [Lévy, 2002; Aradi, 2013], al-Aḥdab's commentary on Ibn al-Bannā's algebra (see below), Moṭoṭ's

[44]Though the general word "polyhedron" is used, it seems that Alfonso actually intends to deal here with parallelepipeds or even strictly boxes. However, it is easy to generalize the proposition to prisms.

[45]This proposition appears to be a spatial version of *Elements* VI.25: "to construct a figure similar to one given rectilinear figure and equal to another." It may also be regarded as a generalization of the Delian problem, where the first polyhedron is any parallelepiped with a volume of two cubic units, and the second polyhedron is a cube whose side is one unit. From another point of view, it may also be regarded as a generalization of Euclid XI.27: "to describe a parallelepipedal solid similar and similarly situated to a given parallelepipedal solid on a given straight line."

[46]Alfonso does not give any details about the way this construction should be carried out.

[47]By the previous proposition.

[48]NM is such that $NM : EG = LN : IG$ (*Elements* VI.12).

[49]$GK : LN = (LN : IG)^2 = (NM : EG)^2 = \text{area}(MO) : \text{area}(EH)$ by *Elements* VI.20. Hence $GK \cdot \text{area}\ (EH) = LN \cdot \text{area}\ (MO)$.

algebra written in Italy (see below), and Finzi's translations of the Italian algebra of Maestro Dardi [Wagner, 2013] and of the Arabic algebra of Abu Kāmil's (possibly through a Spanish or Hebrew middleman) [Levey, 1966].

But even before the "official" algebra using the Khwārizmian terms (root/thing, square/property, cube, etc.) and six normal forms of linear and quadratic equations, quadratic problems were treated by methods that go back to Mesopotamia, namely, the reduction of problems to deriving the values of two unknown from their product and sum/difference with an implicit or explicit geometric model. The following selection opens with problems of the latter type. Then we bring extracts from Moṭoṭ and al-Aḥdab expounding the double false position, classical Khwārizmian algebra, and operations with combinations of powers of unknowns—the forerunners of modern polynomials.

1. QUADRATIC WORD PROBLEMS

We begin with some word problems that involve quadratic equations but that do not use explicit algebraic terms or methods. The basic technique is deriving the value of two unknowns with a given product and sum/difference.

a. A quadratic problem from Levi ben Gershon's *Maʾase Ḥoshev*

Problem 16 of Levi ben Gershon's *Maʾase Ḥoshev* (see section I-6) is a quadratic problem, whose roots can be traced as far back as ancient Mesopotamia.[1] In Mesopotamia, this was a geometric problem, but it also occurs as a purely arithmetic problem in Diophantus's *Arithmetica*, Book I, #27 [Heath, 1964, p. 140]. Diophantus, however, insists on a rational solution, while Levi does not.

16. We multiply one number by another and get the result. The sum of the two numbers is given. What are each of the numbers?

Take the square of half the sum of the two numbers, and subtract the result from it. Take the square root of what remains, and add it to half the sum of the two numbers, to get the first number. If we subtract it from this half, you get the second number.

For example, the sum of two numbers is 13, and their product is 17. We know that the square of half of 13 is 42 and a quarter. Subtract 17, leaving 25 and a quarter. Extract the square root to get 5 wholes and one first, 29 seconds, 46, 34. Add this to 6 and a half, which is half of 13, to get the first number: 11 wholes, 31 firsts, 29, 46, 34. The second number is: one whole, 28 firsts, 30, 13, 26. The product of one with the other is 17 to a very close approximation.[2]

It is impossible to find this number exactly, because 25 and a quarter does not have a true square root, as was explained. This is because the ratio of 25 and a quarter to 25 equals the ratio of one hundred and one to one hundred. But the ratio of one hundred and one to one hundred is not equal to the ratio of a square to a square, since if this were the case, one hundred and one would be a square, because one hundred is a square. But if one hundred and one were a square, then its square root would be a whole number, and that is false.

[1] This problem only occurs in the first edition of the *Maʾase Ḥoshev*. Levi eliminated it in the second edition.
[2] Levi is using sexagesimal fractions, as was taught in part II of his treatise.

If the problem was: we multiply a given number by a fixed part of itself; and we add the result to the product of this part with the remaining part of the given number; and the answer is given; what are each of the parts?[3]

Take the square of the whole number, and subtract from it the sum composed of the product of the number with a part of itself and the product of this one part with the other part. Take the square root of what remains, and this is one part. What remains from the number is the fixed part.

For example, the product of ten with a given part of itself, plus the product of this part with the second part, equals eighty. We want to know: what is the given part? The square of ten is one hundred. We subtract eighty from this to get twenty. We extract the square root, which is approximately 4 wholes, 28, 19, 41, 21, and this is one part. What remains is 5 wholes, 31, 40, 18, 39, which is the given part. If you multiply this part by ten and by the leftover, you get eighty to a very close approximation.

b. Quadratic word problems from an anonymous arithmetic

This section was prepared by Naomi Aradi

An anonymous arithmetic textbook, which survives only partially in three manuscripts, reveals some calculation methods that are seemingly uncommon in medieval Hebrew arithmetic, such as exceptional root approximations. This treatise does not deal with written calculations but only with mental calculations. It opens with an introduction discussing some special properties of the numbers one to ten and general properties of numbers. In the Geneva manuscript excerpted below, a section containing a discussion of algebraic equations was inserted into the introduction. This section was identified by [Lévy, 2002] as an adaptation of a paragraph from the *Algebra* of al-Khwārizmī. After the introduction come six chapters on addition, subtraction, multiplication, division, ratios, and roots. Apparently two additional chapters appeared before the last chapter (on roots): "Deducing (*hoṣaʾat*) One from Another" and "Converting (*hashavat*) One to the Other." In each of these eight chapters the arithmetic operations are presented with integers, sexagesimal fractions, and simple fractions. A detailed outline of the text can be found in [Aradi, 2013].

The chapter titled "Deducing One from Another," which was preserved only in this Geneva manuscript, is devoted mainly to illustrating ways of solving word problems [Aradi, 2013, pp. 277–292]. These problems are presented with short, unmotivated solutions. They begin with a string of highly standardized commercial problems (applications of the Rule of Three to pricing, salary, and partnership) taken from Abraham Bar Ḥiyya's *Foundations of Wisdom*. Then, however, follow problems that are exceptional in that they involve non-homogeneous operations, have no commercial application, and lead to quadratic problems, which are infrequent in Hebrew arithmetic writings. Again, the following problems are not solved by reduction to Khwārizmian normal forms as presented in the Moṭoṭ selection below.

[3]The problem can be formulated as follows: given $A = x + y$, and $B = Ax + xy$, find the parts x and y. The solution is based on the observation that $y^2 = A^2 - B$.

A pricing problem

If you give an unknown [quantity of] *kors* [a Talmudic unit of dry measurement] for a price of 60 and you subtract the price of one *kor* from all the *kors*, 4 would remain.[4]

Take half of the 4, and multiply it by itself, which are 4. Add them to the sixty, and take the root, which is 8. Add the 2, these are 10, which is [the number of] the *kors*. Or, if you subtract the 2 from the 8, six remain, which is the cost of a *kor*.[5]

A salary problem

You hired a salaried worker for an unknown [number of] days for an unknown [quantity of] dinars. When you add up the days and dinars they sum up to 40. He worked unknown days so that when multiplied by their [sic] salary the total is 12. If you multiply the remaining days by the remaining salary, the total is 192. How much are the unknown days, how much are the dinars, and how much is the [payment for] 10 days?

Divide 192 by 12, yielding 16. Take their root, which is 4. Add one, which is 5. Divide 40 by them, 8 comes out. Take their half, 4, multiply them by themselves, and subtract 12 from the result, 4 remain. Take their root, which is 2, add it to the 4, which is half the 8, and the result, 6, is the unknown [number of] days he worked. Alternatively, find the ratio of the 12 to 192, which is half an eighth. Take the root, which is a quarter, and always add one, which are one and a quarter. Divide 40 by them, the result is 32. Take their half, which are 16, and multiply by themselves, yielding 256. Subtract from these the 192, 64 will remain. Take their root, which is 8, and add it to the 16, these are 24. Add them to the first 6, and the sum, which is 30, is the unknown [number of] days of the job. Subtract them from the 40, and the remainder, which is 10, are the unknown dinars.[6]

A partnership problem

One [gave] 10, the second [gave] 20 and the third [gave] 40. When you multiply the profit of the first and the second by the profit of the third, the result is 48. What was the profit of each one?

Add the 10 and the 20, and multiply the result by the 40. The result is a thousand and two hundred. Find the ratio of 48 to the above, which is a fifth of a fifth. Consider them as fractions, and take their root, which is a fifth. Take a fifth as the profit rate for each, so the first is 2, and the second 4 and the third 8.[7]

[4] Note that we subtract a number designating a price from a number designating volume.

[5] Let a be the number of *kors*, and b the price of a *kor*. ab is 60 and $a - b = 4$. The solution proceeds in the standard manner.

[6] If a and b are the number of days worked and salary received, respectively, and a' and b' are the remaining days and salary, we get $a:a' = b:b'$. Therefore, $\frac{a'b'}{ab} = \frac{192}{12} = 16$ is the square of $\frac{a'}{a} = \frac{b'}{b} = 4$. Next, given this ratio, $a + b = \frac{a'+b'+a+b}{4+1} = \frac{40}{5} = 8$. We now have $ab = 12$, $a + b = 8$, which is solved in the standard manner. The alternative solution derives $a' + b'$ from $\frac{ab}{a'b'}$ according to the same procedure. Note that the products and sums are not homogeneous quantities.

[7] Let x be the profit rate. We have: $(10x + 20x)\,40x = 48$, yielding $x^2 = \frac{48}{(10+20)40} = \frac{1}{25}$. Note that the ratio 1:25 has to be "considered as a fraction" for its root to be taken.

c. A quadratic problem from Elijah Mizraḥi's *Book of Number*

This section was prepared by Stela Segev

This problem from Elijah Mizraḥi's *Book of Number* (see section I-5) discusses a quadratic equation and is solved by what we now call Viète's formulas. Elsewhere (problems 50–52) Mizraḥi treats the three kinds of quadratic equations with the standard formulas. In both cases, this treatment remains unmotivated.

Question [24]: If you want to know a number which, when its third is multiplied by its quarter (or any other part of it by any other part), the result is the number itself added to seven, for example, or a multiple of the number added to another number, how can one find this number?[8]

The answer is that this question is composed of multiplication, ratio and addition. Therefore we multiply the two fractions without reference to the sought number and apply ratios to the result. We say: if the result equals some multiple of the sought number added to some number according to the question, how much is one whole?[9] We save the resulting multiple [*mnb*]. We then take the resulting number [*mnc*], and look for all the pairs of numbers which, when multiplied, yield that number.[10] We write them down pair by pair, that is, each pair of numbers, whose product is that number, side by side, each with its partner. The number which, when added to the saved multiple, equals its partner—its sum with the saved multiple is the sought number.[11] That holds when the [sought] numbers are whole. But if [the sought number is] a whole number and a fraction, we look for two numbers that bound the sought number between them, and find the sought number easily because it is bounded between the two numbers.

Example with integers: If you ask which number, when its half is multiplied by its quarter, equals the number itself added to 6, you have to multiply the half by the quarter without reference to the sought number, and you get one eighth. Then we apply ratios and say: if the eighth equals the number itself added to 6, how much is one whole? We get 8 [multiples of the sought number] and the number 48.

We save the eight, and look for numbers which, when multiplied, yield forty-eight. These are pairs, each along with its partner: 1 with 48, 2 with 24, 3 with 16, 4 with 12, and 6 with 8—for when any of these pairs is multiplied, they yield 48. Then we check every pair to see which number, when added to the saved [multiple] 8, equals its partner. We find the number 4 which, when added to the saved 8, yields 12. The sum equals its partner, as the partner of 4 is 12. Therefore we assert that the sought number is 12. From this you will be able to find the solution with integers and fractions as well.

2. SIMON MOṬOṬ, *ALGEBRA*

Our knowledge of Simon Moṭoṭ (or, perhaps Miṭoṭ, i.e., "from Ṭoṭ") comes only from the manuscripts of his two mathematical treatises, one on algebra, and the other on the asymptotic

[8] If x is a number, we can write the equation as: $\frac{x}{m} \cdot \frac{x}{n} = bx + c$.

[9] $x^2 = (mnb)\, x + (mnc)$.

[10] This refers to all pairs p, q such that $pq = mnc$.

[11] When we find a pair such that $p + mnb = q$, then $p + mnb$ is the sought number. This method depends on what we now call Viète's formulas for quadratic equations.

property of the hyperbola, but the latter attribution has been questioned by [Lévy, 1989b]. From the dedication of the algebra treatise, we can date his work to the mid-fifteenth century, and place him in contact with the avid copyist and translator of scientific texts, Mordekhai Finzi [Steinschneider, 1893–1901, p. 193; Lévy, 2007].

Moṭoṭ's treatise of algebra belongs to the Italian abbacus tradition—the culture developed around schools where children of merchants were taught arithmetic. Indeed, Moṭoṭ acknowledges that his sources are Christian, and the content fits well. All the problems and proofs that Moṭoṭ presents appear in one way or another already in the algebraic treatises of fourteenth-century abbacus masters [Høyrup, 2007, pp. 147–182]. (See also section III in Chapter 1.) As was often done at the time, Moṭoṭ opens with a brief definition of algebraic terms (the thing, square, cube, and square-square, corresponding to today's x, x^2, x^3, and x^4, respectively), explains how to make arithmetic calculations with roots and with binomials containing roots, and then presents the six standard kinds of first and second order equations (in modern transcription: $ax = b$, $ax^2 = b$, $ax^2 = bx$, $ax^2 = bx + c$, $bx = ax^2 + c$, $c = ax^2 + bx$) and culminates with some higher order equations that are reducible to them. He accompanies some of his discussions with geometric proofs and numerical examples.

Moṭoṭ claims that the incompleteness of his sources forced him to make up some contents, but it is not clear what his original ingredients were. Perhaps the specific manuscripts consulted by Moṭoṭ were missing some proofs, which he had to reconstruct from his own memory and ingenuity.

There are, indeed, some idiosyncrasies in Moṭoṭ's work, which may indicate slightly different paths of transmission with respect to the dominant algebraic culture. First, the order of presentation diverges from the order used in all early Italian algebras (Moṭoṭ exchanges the fourth and sixth case; see [Høyrup, 2007, p. 160]). The solutions of the first and third examples also stand out as atypical in abacus culture. The fact that the fifth case, which may have no solutions, one solution, or two solutions, is treated as if it always had one, suggests that Moṭoṭ's understanding of algebra was probably not state of the art for his time. Here we bring only the introduction and the treatment of the first six cases.

After praising the Lord, whose renown is glorious and illuminates all utterance and action, may his name be blessed and much exalted, I begin and say.

You should know that in the calculus of algebra [lit.: alzibra] the Christians take one part of a question, whose numerical value is unknown, and turn it in their reckoning into a single whole thing, and call it *cosa*. They wish to indicate by this word two things: one whole thing, and a hidden thing that we do not know. I shall follow suit as well in this translation, and call it by the name thing [*davar*]. The product of the thing with itself they call *censo*. I asked the grammarians of their tongue for the meaning of this word, and they said that it designates a definite number, meaning an unknown definite number. And as we found no single word in our tongue for this meaning, and as I did not want to extend my speech by referring to this meaning by two words, nor to invent a new word in our language, I called it a square [*meruba*ʿ], as it is indeed. And the product of the square with itself they call *censo de censo*, and I call it square of the square. And the cube [*me*ʿ*uqav*] number they call *cubo*. And the cube of the cube they call *cubo de cubo*. The units of numbers they called *numeri*, as is their common habit elsewhere.

The discussion of arithmetic with roots is omitted here.

Now, in the name of He who is known among the nations as the creator, I shall begin to discuss the theorems of the calculus of algebra and explain them with my meager intelligence. But before I begin, I present a proposition and its explanation.

I say: you should learn and have in mind that the ratio of the square-square to the cube is as the ratio of the cube to the square and as the ratio of the square to the thing and as the ratio of the thing to the unit. This is because the number of units in the thing is as the number of things in the square and the number of squares in the cube and the number of cubes in the square-square. Remember this proposition, because you will need it in the proofs of the following theorems. Here I begin.

[1] When the things equal units, divide the units by the things and the outcome is the thing. This is self-evident.

Question: I wish to divide the number ten into two parts, such that when one is divided by the other, the quotient is 5.

Practice this method. Say that the part by which one divides is a thing. The part that one divides is necessarily five things, like the outcome of the division.[12] Added together the two parts are six things, and are equal to the number ten. According to the method indicated in this theorem, the number [10] should be divided by 6, which comes to 1 and 2 thirds. Such is the thing.

[2] When the squares equal units, divide the units by the squares and the root of the outcome is the thing.

Question: I wish to find a number such that when its third is subtracted, the square of the remainder is the number 20.

Practice this method. Say that the number, whose two thirds are the root of 20, is one thing. Multiply its 2 thirds by themselves, making 4 ninths the square of the entire number that I wanted to find. According to the method indicated in this theorem, the number 20 should be divided by 4 ninths, and the outcome is 45. Such is the square of the entire number, and its root is what you wanted.

[3] When the squares equal things, divide the things by the squares and the quotient is the thing.

This theorem follows the first theorem because the ratio of the square to the thing is as the ratio of the thing to the unit, as we said in the proposition. Therefore, if one square equals 3 things, for example, then one thing will necessarily equal 3 units.

Question: I wish to find a number such that when a third is subtracted, the remainder is the root of the entire number.

Practice this method. Say 2 thirds of this number is one thing.[13] Therefore, the entire number is one thing and a half. So one thing and a half equal one square. According to the indicated method, 1 and a half should be divided by one, yielding 1 and a half. This is the thing, which is two thirds of the number you wish to find. The entire number is therefore 2 and a quarter.

[12] In Italian algebra, the parts would usually be modeled as x and $10 - x$, which yields a slightly more complicated procedure.

[13] Note the unusual choice of the thing: x is the required number *after* its $\frac{1}{3}$ is subtracted. Therefore, the entire number is $\frac{3}{2}x$, and we have $x = \sqrt{\frac{3}{2}x}$, or $x^2 = \frac{3}{2}x$.

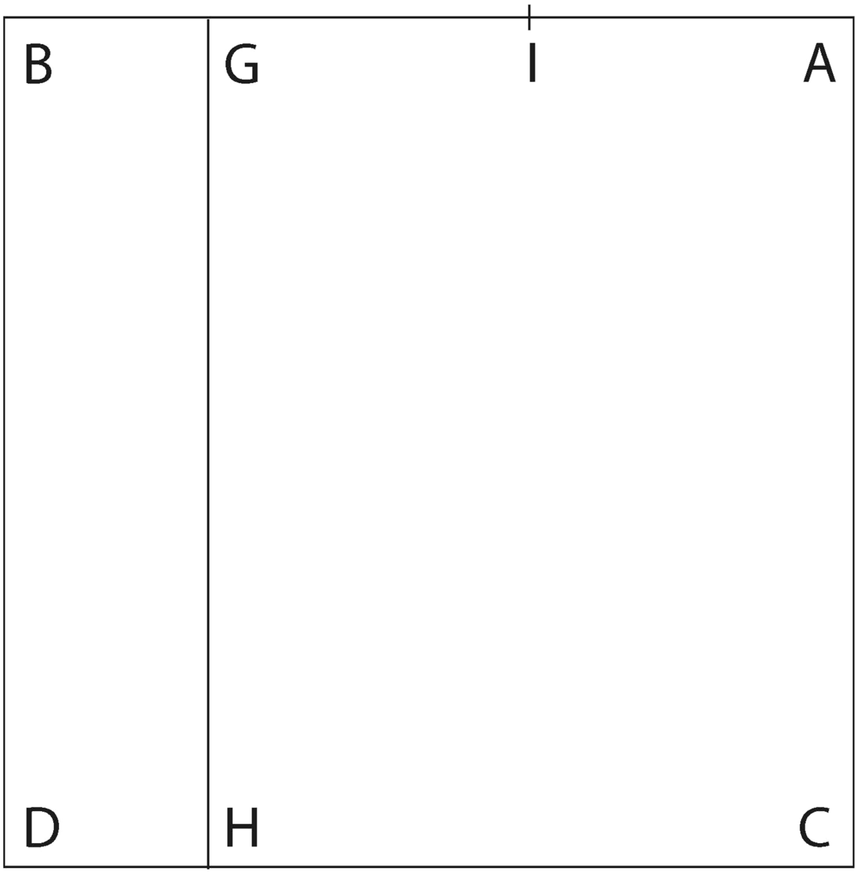


Fig. V-2-1.

[4] When the things and units equal squares, divide the things and units by the squares. Halve the things coming from the division, and multiply this half by itself. Add the result to the units coming from the division. Take the root of the result, and add to half the things coming from the division. The result is the thing.

To demonstrate this [lit.: to show you this for the eye of the intellect], we draw a diagram and bring a numerical example. Let the line *AB* measure 10, and divide arbitrarily at *G*. Let *AG* measure 8. We set the square *ABCD* on *AB*. From the point *G* we draw a line *GH* parallel to the lines *AC* and *BD* [Fig. V-2-1]. We have the surface *AH*, which is eight things (like the measure of the line *AG* in number, because each unit measure in *AG* holds one thing in the surface *AH*) and the surface *GD*, which measures 20 in area together equal the square *AD*. Now here we face the line *AG* which measures 8 in length, like the number of things. We divide it in half at *I*, and add the line *GB*. It has already been shown in the sixth diagram of Euclid's second book that [the square of *IG*, half the line, and] the rectangle contained by the entire line with the addition [*AB*] and the addition [*GB*] (which equals the surface *GD*, whose area is the number 20 in our example) together, being 36, equal the square of the line composed of half the line [*IG*] and the addition [*GB*], which is the line *IB* in our diagram. Therefore, if you take the root of 36, which is 6, you get the measure of the line composed of half the line and the addition, which is the line *IB*. Add half the things, which is the number 4, as the measure of the line *AI* in number, and 10 will result as the entire line *AB*, the side of the square. This is the thing.

Question: We wish to find a number such that adding to it 28, it will equal two times its square.

Practice this method. Say that this number is one thing. When we've added 28 it becomes one thing and 28 units. These equal two squares. Then, according to the method indicated in this theorem, one thing and 28 units should be divided by 2, the number of squares. The quotient is half a thing and 14 units. Take half of the half thing, which is the quotient. It is a quarter of a thing. Multiply it by itself, it is one part in 16. Add to 14, the number of units in the quotient, yielding 14 and one part in 16. Take its root, which is 3 and 3 quarters. Add it to half the things in the quotient, which is a quarter of a thing, yielding 4. This is the thing.

[5] When the squares and units equal things, divide the things and the units by the squares. Halve the quotient of the things and multiply by itself. Subtract from the result the number which is the quotient of the units. Add the root of the remainder to half the quotient of the things. The result is the thing.

The geometric proof is omitted here.

Question: A merchant went trading with a certain capital, and earned 6. He then returned with that capital and the profit, and made a profit at the same ratio as in the first round, having altogether 27. You wish to know the number of the initial amount.

Practice this method. Say that the initial capital is one thing. He succeeded and made this thing into a thing and 6, and by the same ratio, from one thing and 6 he made 27. The ratio of a thing to a thing and 6 is as the ratio of a thing and 6 to 27 units. We have three magnitudes in proportion. It is known then from proposition 17 of the sixth book of Euclid that multiplying the first by the last equals multiplying the middle by its own image. Now multiply one thing, which is the first, by 27 units, which is the last, yielding 27 things. Then multiply one thing and 6, which is the middle, by itself, yielding one square and 12 things and 36 units. Now subtract the 12 things from these two equal magnitudes, leaving 15 [things] equal to one square and 36 units. According to the method we stated in this theorem, the number of things, 15, and the number of units, 36, should be divided by one, which is the number of the square. The quotient is 15 things and 36 units. Then halve the things, which are 7 and a half, and multiply by themselves. The result is 56 and a quarter. Subtract 36 units, leaving 20 and a quarter. Take its root, which is 4 and a half, and add to half the things, which is 7 and a half, yielding 12. This is the thing which is the initial capital.

[6] When the squares and things equal units, divide the things and the units by the squares. Halve the quotient of the things and multiply the half by itself. Add the result to the quotient of the units. The root of the result less half the things in the quotient is the thing.

The geometric proof is omitted here. This case is not accompanied by a numerical problem.

This is the span of what I sought and found here and there about the calculus of the book of algebra in the books of the Christians. I made up many of these theorems myself. You should know, my dear brother Mordekhai (that he might see offspring and have long life, and that through him the Lord's purpose might prosper),[14] son of our honorable master Abraham Finzi (may his memory live in the world to come) that the author of the book of all these theorems brought them in his book without proofs, and none of those who read it knows the methods of this scholar and where he found them. I, your brother, seeing you and my dear friend Rabbi Judah, son of our honorable master Joseph (may God save him and keep him alive), son of our honorable master Avigdor (may his memory live in the world to come), longing to know it, and, as he who knows, if we are to call him "one who knows," must know by logical proof—I had to study the proofs and write them for you so as to fulfill your wish.

I was, however, succinct for two reasons. The first is because I trust your good spirit, the divine spirit hovering over all wisdom. The second is the toil and trouble that came upon me to preoccupy my mind and body, and my many dealings in worldly business. But if one of you misses anything due to my brevity and weariness of long proofs, I say that I am willing to further clarify it. One must not be long, except in appealing to God, may He fulfill all thy wishes, let thy springs spread, springs of salvation, Amen. In accordance with your will and the will of your faithful brother, who abides by your command, Simon, son of our honorable master Moses (may God save him and keep him alive), son of our honorable master Simon Moṭoṭ (may his memory live in the world to come).

3. IBN AL-AḤDAB, *IGERET HAMISPAR* (*THE EPISTLE OF THE NUMBER*)

This section was prepared by Ilana Wartenberg

Isaac ben Solomon ibn al-Aḥdab (Castile, ca. 1350–Sicily, ca. 1430) was a Jewish polymath. His prolific writings covered a wide range of fields: astronomy, reckoning of the Jewish calendar, mathematics, exegesis, and poetry. After leaving Castile, he studied with Muslim scholars in North Africa. Then, on the way to the Holy Land, he was shipwrecked in Syracuse, Sicily. At the request of the local Jewish community there, Isaac composed *The Epistle of the Number*.[15]

The Epistle of the Number is the first and only known Hebrew version of the succinct Arabic mathematical tract *Talkhīṣ Aʾmāl al-Ḥisāb* (*A Summary Account of the Operations of Calculation*) written by the famous Moroccan mathematician Aḥmad ibn al-Bannāʾ (1256–1321) (see parts I-1 and II-1 of Chapter 3). *The Epistle of the Number* is not only the first Hebrew text we know of that contains explicit algebraic materials but it also includes numerous arithmetical themes.

The Epistle of the Number presents a perfect translation of *Talkhīṣ Aʾmāl al-Ḥisāb* as well as detailed mathematical explanations accompanied by a multitude of numerical examples, philological and philosophical discussions. (See section II-1 in Chapter 3 for a direct translation of this part of *Talkhīṣ* from the original Arabic.) The Hebrew text follows the structure of its main Arabic source: the first part is dedicated to arithmetic, that is, arithmetical operations on three types of known quantities (numbers): integers, fractions, and roots. The second part of the book presents three methods to determine the value of

[14] Isaiah 53:10, quoted in acronym.

[15] *The Epistle of the Number* survives in a fragmentary unicum [Cambridge University Library, Heb. Add. 492.1, ff. 1b–38b]. For various analyses of the text and its context, the author's life, and a critical edition and English translation, see [Lévy, 2003; Wartenberg, 2007, 2008a,b,c, 2013, 2014].

the unknown: the Rule of Three, the rule of false position, and algebra. The part on algebra includes lengthy discussions of algebraic elementary entities (numbers, roots, and squares) and algebraic operations (restoration, opposition, and equation). It also presents a detailed analysis of algebraic expressions, the "ancestors" of modern polynomials. At the very end of the truncated unicum we find a series of various problems (e.g., charity distribution, time-velocity-distance calculations) that are solved by algebraic methods.

Double false position

The method of scales is usually referred to in modern mathematics as the rule of false position. This method was widely known in the Arabic mathematical tradition.[16] It is an arithmetical procedure that enables one to find the value of the unknown in what could be anachronistically termed "linear" problems by balancing out the errors of wrong guesses. The standard method of double false position requires two different guesses, and it can be used in problems with two unknowns. However, where there is only a single unknown, the method can be adapted to a single guess. Both possibilities are treated below.

The basic tool used in this method are scales, where one writes one's guesses, the results of substituting these guesses into the problem, and the resulting errors. Since the errors are considered as absolute numbers (rather than signed numbers), the algorithm depends on the direction of errors. Ibn al-Aḥdab treats all such possibilities, but here we only include an example where the errors are positive.

He [the author of the *Talkhīṣ*] says: **The method of scales is part of the art of mathematics**[17] **and their shape is to be drawn as follows** [Fig. V-3-1]:

Place the given known number on the fulcrum. Take one of the pans, and then take any number you wish and carry out the procedures which were given, whether addition or subtraction or another procedure. Then compare the result with the number on the fulcrum. If you find that it is the same value, then the value you have chosen is the unknown number.

He says that this is part of the science of mathematics because at times he adds, at times he subtracts, and takes the intermediate value and this is found in mathematics such as the mean, minimal and maximal distances found in [astronomical] tables, and so on.

You already recognize from the form of the fulcrum that it is the upper part of the scales, which is called the *level*, and the pans are hung by threads. Draw it as follows [Fig. V-3-2]: or in any form you wish.

We shall give the example we have given in the part on the proportional numbers,[18] i.e., a wealth [that is, a certain amount of money], from which one has subtracted both a

[16] According to [Suter, 1901, p. 31], the method of scales was already known in Baghdad in the ninth century at the time of al-Khwārizmī; it was also widely used in North Africa and in the Middle East. See [Youschkevitch, 1976, pp. 45–48]. The method of double false position appears earlier in the Chinese work *Nine Chapters of the Mathematical Art* (first century BCE) [Dauben, 2007, pp. 269–274]. See section II-3-2 in Chapter 1 for its appearance in Fibonacci's *Liber abbaci*.

[17] The Hebrew term *hamelakhot halimudiyot* usually refers to astronomy or mathematics, the latter being most fitting in our context. Here it is used to translate the Arabic term صناعة الهندسية, which literally means "the art of geometry." See Section II-1 in Chapter 3.

[18] This refers to the Rule of Three, which was explained right before the method of scales.

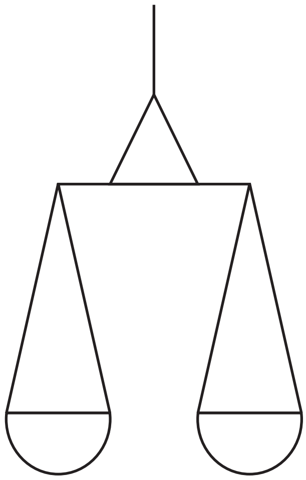


Fig. V-3-1.

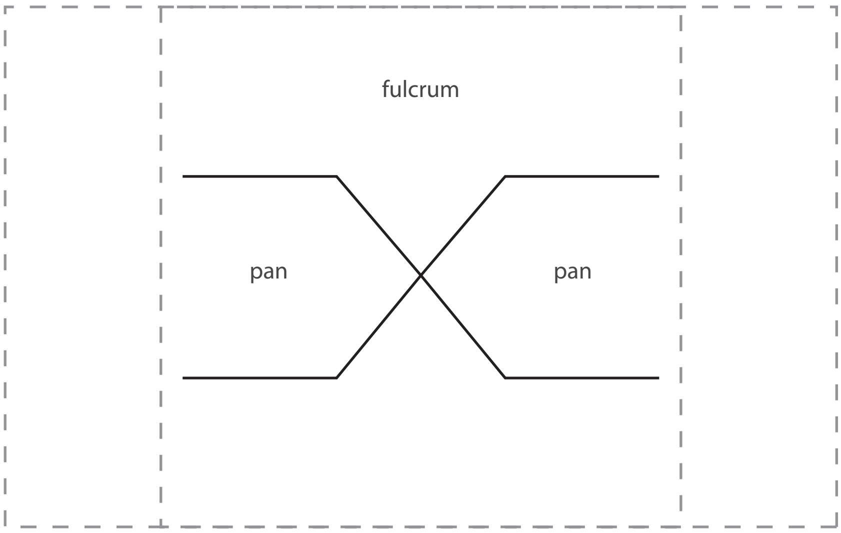


Fig. V-3-2.

third and a quarter, and ten is left. How large is the wealth?[19] Draw the scales as follows [Fig. V-3-3]:

and write ten on the fulcrum, which is the given known in the problem. Then take whichever number you wish, write it in one of the pans, and proceed according to what is

[19]If we use anachronistic notation and denote the unknown wealth by x, the problem is: $x - \frac{1}{3}x - \frac{1}{4}x = 10$. We will use the notation: $f(x) = x - \frac{1}{3}x - \frac{1}{4}x$. Note that here the operation is linear. In general, the method works also for affine functions, where $f(x) = Ax + B$.

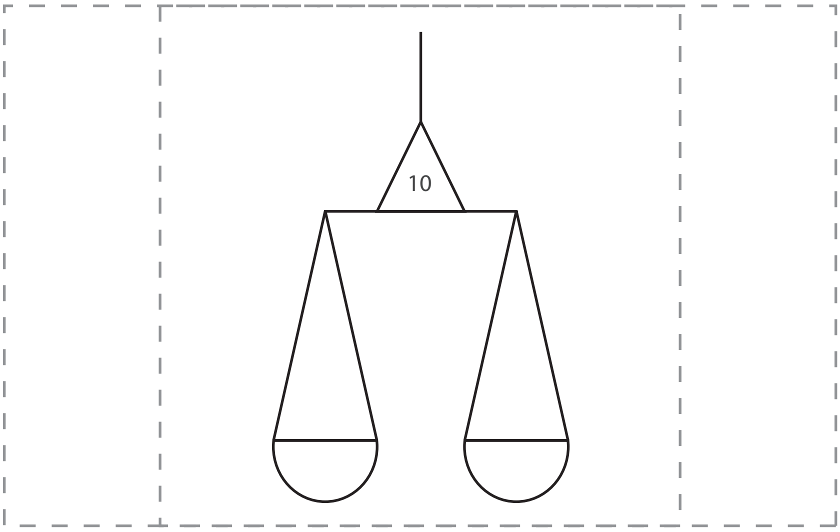


Fig. V-3-3.

mentioned in the problem, by subtracting its third and its quarter and taking the rest, comparing it with the ten on the fulcrum, i.e., inspect whether it is the same value, less or more.

. . .

He says: **If you err then write the error above the pan if it is superfluous, or underneath, if it is deficient. Then place in the other pan any number you wish, except for the first one you have chosen; follow the same procedure as you have done in the first case. Then multiply the one error by the other integer. Inspect whether the errors are superfluous or deficient. Subtract the smaller error from the larger one and subtract the smaller multiplication from the larger one. Divide the error between the multiplications by the remainder of the errors.**

Commentary: if it did not occur that you have chosen the right number, but a different one, which is the case most of the time—the other case [choosing the right number] is rare—in any case, after you subtract one third and one quarter, the error in relation to the ten on the fulcrum, will be either larger or smaller than it; this is what he meant by "if you err."

For example, you draw the scales as follows; take, for example, the number 36, write it in one pan, subtract its third, 12, and its quarter, 9, altogether 21, and the remainder is 15. Compare it to the ten and there is an error, because there are five superfluous ones, i.e., units. Write the five, which is the error, above the pan, because it is superfluous. If the remainder were 8, being less than 10, you would write it beneath the pan. Then place any number you wish on the other pan, except for the 36 that you have already chosen, e.g., 48. Write it in the other pan, subtract its third, 16, and its quarter, 12, altogether 28, the remainder is 20. Compare it to the ten on the fulcrum, there is an error here too, because

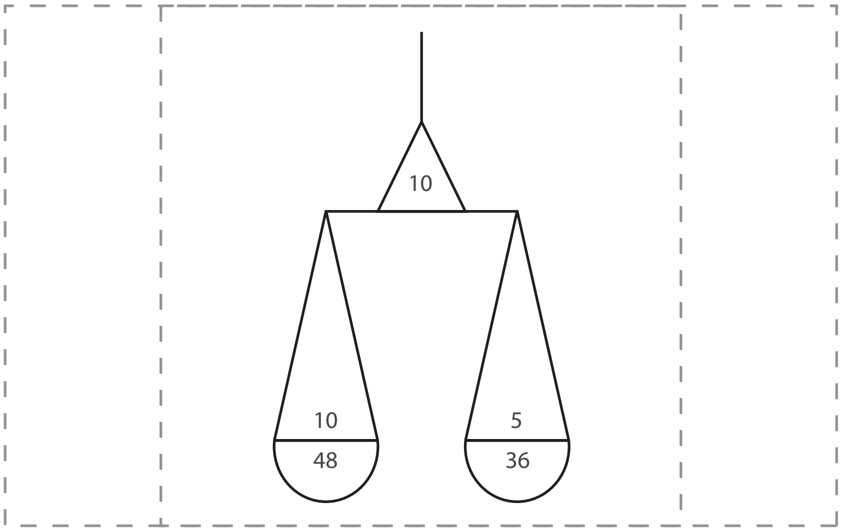


Fig. V-3-4.

there are ten superfluous ones. Write it above the pan because it is also superfluous [Fig. V-3-4].[20]

. . .

Then multiply each error in each pan by the integer in the other, i.e., multiply the 5 above the 36, which is the error in the first pan, by the integer in the other pan, 48, the result is 240. Also, multiply the ten above the second pan, which is its error, by the other, it becomes 360. 240 and 360 are called *multiplications*. This is the first procedure for the case when the errors are above the pans.[21]

. . .

He said: **Then inspect whether both errors are larger** than the ten in the pan, i.e., such as in the first example, **or deficient**, i.e., like in the second example.[22] **Then subtract the smaller from the larger**, i.e., the smaller of errors from the larger one, i.e., subtract the smaller error, which is 5 in the first example, from ten, which is above the second pan, and 5 remains. The **smaller of multiplications** refers to subtracting the smaller multiplication, such as 240 in the first example, **from the larger one**, which is 360, and

[20] $f(36) = 15$ and $f(48) = 20$; $f(36) - 10 = 5$ and $f(48) - 10 = 10$.
[21] $48 \times (f(36) - 10) = 48 \times 5 = 240$; $36 \times (f(48) - 10) = 36 \times 10 = 360$.
[22] This second example is omitted here.

the remainder is 120. **Divide the remaining part of the multiplication**, which equals 120, **by** 5, **the remainder of the errors.** The result, 24, is the unknown sought number.[23]

. . .

He said: **If one of them is superfluous and the other is deficient, divide the sum of the multiplications by the sum of the errors.**[24]

. . .

He says: **If you wish, place in the second pan the first number or another one, and subtract from it the part which is compared to what is above the fulcrum, then multiply it by the integer in the first pan and multiply the error of the first by the integer in the second pan. Then, if the error of the first pan is deficient, add the multiplications. If it is larger, take the difference between them and divide it by the part, i.e. the number, in the second pan.**

Commentary: This is another method in the procedure of the scales in which you take whichever number you wish and write in one pan, do with it as you have in the first procedure, write this error above if it is superfluous, or below, if it is deficient.

We demonstrate it with an example: let there be ten on the fulcrum and place 36 in the first pan. We follow the same procedure as before; the error, 5, will be above the pan.

. . .

We then turn to the second pan and write 48 in it as we have done before. . . . If we take 48 and subtract its third and its quarter, twenty is left and it is that term with which we compare. In this procedure, we write the error between this number and the ten above the fulcrum neither above nor below the pan, but we write the integer either above or below as we wish. . . . Multiply this part by the other, which is 20, when the number written in the pan is 48. If the first pan had 36, we would multiply 20 by 36, becoming 720. We multiply 5, which is the error above the first pan, by 48, and it becomes 240 [Fig. V-3-5]. Since the error is above the pan, which is superfluous, we take the increment between the two multiplications, i.e. we subtract 240 from 720, and the remainder is 480. We divide it by 20, which is called *division by what is above the second pan*, and the result is 24; it is the desired solution[25] and this is its form:

[23]The operation here is: $\frac{36\times(f(48)-10)-48\times(f(36)-10)}{(f(48)-10)-(f(36)-10)}$. This number solves the problem $f(x) = 10$. Indeed, the ratio $(48-x):(48-36)$ should be the same as the ratio $(f(48)-f(x)):(f(48)-f(36)) = (f(48)-10):((f(48)-10)-(f(36)-10))$. The value of x can now be derived by means of proportion theory.

[24]Since the author considers the absolute values of the errors $f(a)-10$, the difference in the denominator has to be replaced by a sum when one error is negative and the other is positive.

[25]Here we have the solution $\frac{36\times f(48)-48\times(f(36)-10)}{f(48)}$, which applies when f is linear ($f(x) = Ax$). This solution retains its meaning even if we only have one sampling of f, in which case it reduces to $\frac{48\times 10}{f(48)}$, namely, to a simple application of the Rule of Three in a single false position.

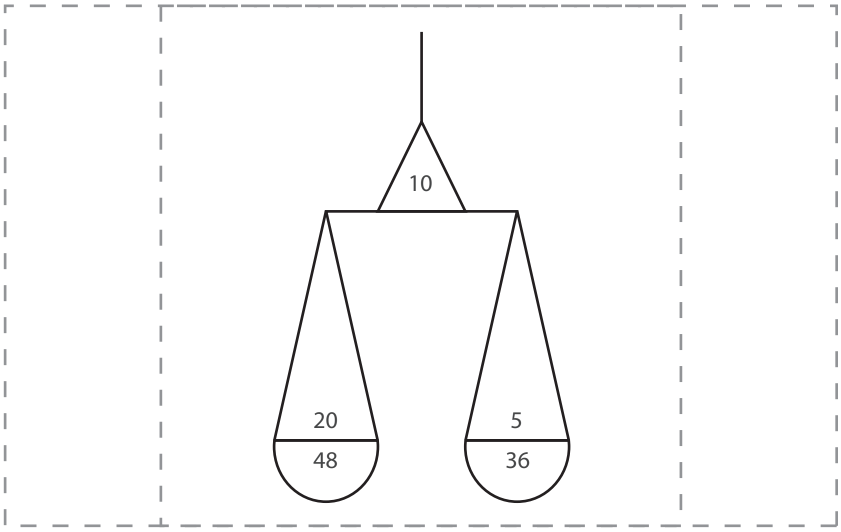


Fig. V-3-5.

The elements of algebra and their operations

Here are presented the algebraic terms (root or thing, square or estate, etc.). These terms are presented as analogous to numbers and decimal ranks, but unknown algebraic terms are carefully distinguished from known numbers, which could be roots and squares as well. The discussion refers to the possibility of unknown algebraic terms that are not determined with respect to each other (a root which is not the root of the given square, that is, referring to a different unknown).

He said: **The application of restoration**[26] **is upon three species: numbers, things, and estates. The things are the roots. The squares are the result of the root multiplied by itself.**

Commentary: As previously explained, the number has three ranks: units, tens, and hundreds and all other [numbers] are composite by them. Also, in this science, ranks are set which are numbers, roots and squares and the rest are formed by them. This is what he meant when he said: the application of restoration is upon three species. Even though there are many species, cubes, and others, here, the main aim of this science is to reduce them all to these three species.

The three species are numbers, things, and estates. He said that the things are the roots previously mentioned in the book, and in this science one names them things. The estates are the result of the root multiplied by itself. In other words, when one multiplies the root by itself, this multiplication is named an estate in this science, and it is the square that was mentioned in the book.

The commentary of these two: he set [aside] the number because these two are also named in this science by a different name, but the number is not. A complete commentary about these three species is as follows: the numbers could be any number, whether units,

[26] "Restoration" is one of the elementary algebraic operations (*al-jabr*), that of adding a subtracted term to both sides of an equation.

tens, or hundreds, other ranks or their composition, either large or small, such as 5, 9, 11, and 120, and in general, every number, either large or small. For this reason this rank is named the number.

The roots are the roots of the squares. It is known that every number can be a root of a square. . . . That number is a root to that square and for every number [it is the] same. However, the number is called a number in itself and it is not called a root unless related to a square. Therefore, the number given here is a number as such. The given root here is a number which is a root extracted from a certain square and it does not have a determined value. That is why it is called a thing, i.e., a certain thing among the numbers that is a root to a certain square.

. . . There is no interest in a square [per se] in the Science of the Number but [rather] in the result of the multiplication of a number by itself. For this reason, the given square here is an unknown square and that's why they named it an estate.

Also, the roots, that are called things, and the squares, that are called estates, are always set in the problems as unknown. [As for] the numbers, there is no way to set them as unknown, but only as known, since the number has no relation to others, such as the root has with the square or the square has with the root, because the number, as I said, is a number in itself, the root is a root to a square, and the square is a square to a root, in the way of all [things which are] related.

This is the explanation of the three species, on which restoration acts.

The hundreds are formed by tens and units and the thousands are formed by hundreds, tens are formed by units, and all the other ranks are formed by previous ranks. Also, in this science, the squares are joined by the multiplication of the root by the root and are called estates. The cube [is formed] by the multiplication of the root by an estate. If one multiplies the root by a cube, it becomes of a different rank, called the estate of an estate, it is the multiplication of an estate by an estate. If one multiplies an estate by a cube, the result becomes of a different rank, called an estate-cube. A cube [multiplied] by a cube becomes a cube-cube, and so on with the rest. The repetitions of estates and cubes are called an estate-estate and a cube-cube-cube or their composition, an estate-estate-cube, an estate-cube-cube and similarly [with] the rest. As numbers have ranks, so do these [objects].

Just as numbers can be added, subtracted, multiplied, and divided, so can numbers, roots, and squares. The other ranks are also added together, subtracted from one another, multiplied and divided by each other. He says: the number, which is 5 and 7 added together, equals what 8 and 4 add up to. He also says that roots are equal to a number, or roots are equal to an estate, or roots and a square [equal to] a number and so on. . . .

There are 4 types of the number,[27] the first of which is an isolated number, such as 5, 30 or others, and it is called integer. Also in this science, we find a root by itself or an estate and the term in which one of these is found is called integer.

[27] The following categorization does not appear in the Talkhīṣ. The four types are: whole terms (e.g., a number, a root, or a square); connected terms (sums whose parts are not related, that is, are not of the same kind and cannot form a ratio—e.g., numbers counting different kinds of objects, or squares and roots of different unknowns); added terms (e.g., sums of numbers of the same kind, roots and squares of the same unknown); and subtracted terms (differences of numbers or algebraic terms).

The second type refers to having two numbers or more, where each stands by itself, such as 4 and 5 or 8 and 12, etc., such that one number is not related to the other, and it is called connected. Similarly, in this science it is said that a term has, for example, a root, a square and a cube, and each one stands by itself; i.e., the roots are not of the squares and the squares are not of the cubes. Rather, each stands by itself and is added to the other and is also called a connected term. This is scarcely encountered in this science, unless within numbers. . . .

The third: when he says, for example, one and a quarter and one and a half, where the quarter is related to one. So is one half, because one quarter is one quarter of the one and this is called the additive, because the part is added to the integer. Also, in this science, it is said that a term has squares and roots, i.e., an estate and things, or a cube and an estate which are related, meaning that the things are things of the estate and the estate is an estate of the cube and it is called additive. Also, an estate plus a number and a thing plus a number, are called additive even though they are not related or connected.

The fourth: as he says, in "number ten minus 2" or "30 minus 5" or "one minus one third," where it is called deficient or subtractive, so it is in this science, it is said that a term has an estate less a thing, or a cube less a square, which is called deficient or subtractive.

Multiplication and division of algebraic terms

The fragment below comes after a systematic treatment of adding and subtracting compound algebraic terms and a discussion of multiplying binomial sums and differences of algebraic terms. Each algebraic term is assigned a degree (1 for root, 2 for square, etc.), and the degree of the product is the sum of the degrees of the multiplied terms. A notation that places the degree above the coefficient (e.g., 2 with superscript 3 for "2 cubes") is introduced but is not actually put to use. These operations are presented as arithmetical operations in their own right, but their discussion is interspersed with applications to simplifying equations (adding and subtracting terms and reducing the degree of an equation through division by the term of least degree).

Whenever we wish to multiply ranks by ranks, the custom is to write them in two lines, one beneath the other. The degrees of the superior line are above and the degrees of the inferior line are beneath. For example, you wish to multiply 3 cubes and 7 estates and ten things by 9 cubes and six estates and 5 things. Write them in this form:

1	2	3
10	7	3
5	6	9
1	2	3

Then, we actually write three lines. In the middle one write the degrees by order, i.e., start by the larger and end by the smaller. In the lower line, write the outcome of the multiplication of the additives, and in the upper [line, write] the outcome of the multiplication of the subtractives, each below the degree which corresponds to it or above it in this form:

0	1	2	3	4	5	6
		50	35 60	15 42 90	18 63	27
		50	95	147	81	27

Start with the degree 6, as it is the biggest in this multiplication. At the [left] end, we write a zero, as this is appropriate when there is a number in the multiplication. Then, multiply 3 by 9, and the outcome is 27. Add the degrees, 6, and this is why we write the 27 under the 6. Then, multiply 3 by 6, and it becomes 18. Its degree is 5, and that is why we write it under the 5. Then, multiply 3 by 5, it becomes 15, and its degree is 4. That is why we write it under the 4. All multiplications of the 3 are hereby ended.

We return to the 7. Multiply it by 9, it becomes 63 and adding its degrees it becomes 5. That is why we write it under the 5. Then multiply 7 by 6. It is 42 and their added degrees are 4. That is why we write it under the 4. Then, multiply 7 by 5, it is 35 and its degree is 3. Write it under the 3 and the multiplication of the 7 is [hereby] ended.

Return to the 10 and multiply it by 9. It is 90 and its degree is 4. That is why we write the 90 under the 4. Multiply 10 by 6, it is 60 and its degree is 3. Write the 60 under the 3, then, multiply 10 by 5. It becomes 50 and its degree is 2. Write the 50 under the 2. Then, add each species with each other and the result is 27 square-square-estates plus 81 square-cubes plus 147 square-estates plus 95 cubes plus 50 estates. Write them each under its own species.

Another example: we wish to multiply 2 estates less 3 things by 3 cubes less 4 estates in this form:

2		1
2 3	Less Less	3 4
3		2

Multiply 2 by 3. They are [the multiplication of] an additive by an additive, or say an integer by an integer. It is 6 and the addition of their degrees is 5. That is why we write it underneath the 5 [see table below]. Then, multiply 2 by less 4, and it becomes subtractive 8. It is subtractive so we write it above the 4, as the addition of their degrees results in 4. Then, multiply less 3 by 3, it becomes subtractive 9 and its degree is 4. That is why we write it above the 4 as well. . . . Then, multiply less 3 by less 4 as if the word less were not there. It is 12 and its degree is 3. That is why we write it under the 3. Then, add each species with its own and the outcome is 6 cube-estates less 17 square-estates and 12 additive cubes. We write them below each other in correspondence to their species.

	9 8	
3	4	5
12		6
12	Less 17	6

. . .

He said: **when one divides a species of these species by its inferior, subtract the degree of the divisor from the degree of the divided term and the rest is the degree of the outcome of the division.**

Commentary: it is already known that the lower of the species that has a degree is the root. Next, the greater [species] is the square, i.e., the estate, and after it the cube, and so on. One knows in the division of integers that the division has two forms: first, the division of a large number by a small number, or express it as "a superior number [divided] by an integer number," and this is called simple division. The second [division] is the division of a small number by a large number and [the former] is called by the denomination [of the latter].

The second [division] is not commonly found in these chapters. One knows this from the condition that the writer set here about the knowledge of the outcome of the division, by subtracting from the degree of the divided term the degree of the divisor. This necessitates that the degree of the divisor is smaller than the degree of the divided term. Also, the author tells us next not to divide the lower of the species by the superior. His instructions are that when dividing integers, it is a number by a number. However, the divided term is [of] superior [degree] and the divisor [is of] inferior [degree].

For example, one wishes to divide ten estates by 5 roots. Divide ten by 5, as with integers, and the outcome of the division is 2. In order to know [of] what [species] this 2 is, subtract the degree of the divisor, roots with degree of 1, from the degree of the divided term, which is estates of degree 2. 1 remains, and it is the degree of roots. Here, the 2, the result of the division, is 2 roots. Then, if you wish, [you can] test the division from what you already know, by multiplying the result by the divisor, and the outcome is the divided term. Also, in this example, multiply the outcome, 2 roots, by 5 roots, the divisor, and it becomes ten estates with a degree of 2.

. . .

He said: **when dividing one species by the same one, the outcome is a number.**

Commentary: when dividing roots by roots and subtracting the degree of the divisor from the degree of the divided term, nothing remains. Thus, the outcome of the division is a number without a degree. For example, when ten roots are divided by five roots, the outcome of the division is 2 and they are *zuzim*,[28] because the degree of the divisor is 1, and so is the degree of the degree of the divided, a root. When subtracting 1 from 1, nothing remains. . . .

He said: **when dividing one of these species by a number, the outcome is of the same species.**

Commentary: as the number has no degree, when dividing any of these species by it, the divisor, which is the number, has no degree to subtract from the degree of the divided term. Thus, the degree of the divided term remains in its place. For example, when 12 cubes are divided by 4 *zuzim*, the outcome is 3 cubes.

[28] A Hebrew coin denomination.

He said: **if the divided term contains subtractive terms, divide each term, in both the subtractive and the subtrahend, by the divisor, and then subtract the outcomes.**

Commentary: shortly the author mentions that no species containing subtractives can divide. However, the divided term, even though it is subtractive, can be divided. He gives the method here, [i.e.,] dividing the subtractive alone by the divisor and dividing the term subtracted from as well. Then, subtract the outcome of the first from the outcome of the second. The remainder is the answer being sought.

An example here: you wish to divide 4 estates less 6 roots by two roots. Divide the subtractive, 6 roots, by the divisor, 2 roots, and the outcome is 3 *zuzim*. Then, divide the part from which the subtractives are subtracted, 4 estates, by the divisor, which is 2 roots[, producing 2 roots]. One subtracts the first outcome, 3 *zuzim*, from the second outcome, 2 roots, and the final outcome is 2 roots less 3 *zuzim*, which is the result of the division by its divisor, which is 2 roots. . . .

He said: **one shall not divide by a subtractive.**

Commentary: the divided is not subtractive. The reason being is that you already know that the definition of division, according to what the author wrote in the chapter on division, is the decomposition of the divided term into equal parts, and their number is reflected by the divisor in units. I have already explained this definition there. It follows from it that the knowledge of the units of the divisor and all these species in these chapters are unknown. When the species is subtractive, we do not know what is left of its subtracted term for us to divide by this remainder.

For example, if the divisor is a square less 2 roots, when we subtract two roots from the square, then we will not know how many roots are left in a square to divide by it. Similarly in all species, and this is explained.

He said: **This completes what we wished to know. Thank God, may His name be magnified and blessed.**

Commentary: He bestowed praise upon the Almighty, may His name be magnified, who helped him complete his mission as the erudite authors rejoice upon the completion of their composition and they thank the Almighty, may His name be blessed, who favored them and helped them make their name respected. We praise the exalted God of Israel by every blessing and praising, who helped us with its explanation. He will help us with His mercy, with everything of the honor of His name. He will save us. He will tell about our sins for his great and fearful name, may His name be blessed and magnified. Amen.

. . .

The commentator said: I have also seen [it appropriate] to write in this chapter a supplement [written] by their [i.e., Arab] scholars in this science, as I have seen it to be obligatory and useful in all species of restoration.

One already knows what was mentioned about division in [the part about] restoration, [i.e.,] that one does not divide by a subtractive [term] and also, that a lower species does not divide an upper species. It is known that there are instances in the problems where it is required that the lower divide the upper and, also, divide by a subtractive.

Therefore, when this is given in the problem, one has to apply this from the known rule, because when one multiplies the outcome of the division by the divisor, the outcome of the multiplication is the divided term. Therefore, multiply what was given from the result of the division by its divisor. The outcome of the multiplication is equal to the divided term.

The example mentioned in the third problem [that is, split 10 into two numbers, such that one divided by the other is 4], in which he said: when one divides the one part of the ten by the other, the outcome of the division is 4, because, if we let the one part [be] a thing and the other [be] ten less a thing, then this is a division by a subtractive . . . and it does not divide. Indeed, he states that the outcome of the division is 4. Therefore, when multiplying the 4 by ten less a thing, the divisor, the outcome is 40 less 4 things and it equals the divided term. When one completes and opposes,[29] it becomes 40 equals 5 things. When one restores, 8 is obtained, and it is the one part of ten and the other is equal to two.[30] Use it in analogy.

Sometimes, there are instances in such problems, where you are not able to determine the solution by this rule. Therefore, you need to know a different rule. In the example here, in which one asks: Ten, partition it into two parts, divide each one by the other and add the outcome of the two divisions. The sum is 2 and one sixth.[31]

If one lets the one part [be] a thing and the other be ten less a thing, then there is no way to divide by each other, because the one is division of a lower by upper and the second is a division by a subtractive, as mentioned. There is no way to solve it by the mentioned rule because the given in the problem is the outcome of the two divisions, added together. We do not know what will come out of each division in order to multiply it by its divisor, to give the divided term.

Therefore, you need here a different rule. The rule is that for all the numbers [in the problem], divided by one another, i.e., by each other, add the outcomes of the two divisions and keep them. Then, multiply each of those numbers by itself, add the outcomes, and then it becomes the multiplication of one of the numbers by the other and the kept outcome.

An example of this rule: 4 and 6. Divide the six by the 4, and from the division we obtain one and a half. Divide the 4 by the six, the outcome is 4 sixths. Add the latter with the one and half, it becomes 2 and one sixth, which is the outcome of the divisions. Keep it and multiply 4 by itself, and it becomes 16. [Multiply] six by itself, and it becomes 36. Add 16 to 36, and it becomes 52. Also, when multiplying 4 by six, it becomes 24. Multiply it by the kept 2 and one sixth, this is 52. And this equals the sum of the two multiplications, because it is also 52.[32]

After knowing this rule, return to the problem that he set. He says that when one divides the one part of the ten, which is a thing, by the other part, which is ten minus a thing, divide the ten less a thing by the thing, add the outcomes of the two divisions, and the outcome is 2 and one sixth. Do as mentioned in the rule of multiplication. Multiply the thing by itself,

[29] Opposition is the algebraic operation of subtracting equal terms present in both sides of an equation (*al-muqābala*).

[30] The problem is to split 10 into two parts, x and $10 - x$, such that their quotient is 4. Instead of the inadmissible division $\frac{x}{10-x} = 4$, the commentator introduces the equation $x = 4\,(10 - x)$ to obtain $x = 8$.

[31] The problem is: $\frac{x}{10-x} + \frac{10-x}{x} = 2\frac{1}{6}$, which includes inadmissible divisions. To solve it, rearrange it as: $x^2 + (10 - x)^2 = 2\frac{1}{6}x\,(10 - x)$.

[32] $\frac{4}{6} + \frac{6}{4} = 2\frac{1}{6}$. This is equivalent to $4 \times 4 + 6 \times 6 = 4 \times 6 \times 2\frac{1}{6}$.

and it becomes an estate, multiply the ten less a thing by itself, the result is 100 minus 20 things and an estate. Add it with the estate which results from the multiplication of the thing by itself, and it becomes 100 less 20 things plus 2 estates. Then, multiply the ten less a thing by a thing, and obtain 10 things minus an estate. Multiply it by 2 and one sixth, as [previously] mentioned to be the outcome of the two divisions. The outcome of the multiplication is 21 things and two thirds of a thing less 2 estates plus one sixth [of an estate] equals the 100 less 20 things and 2 estates.

When one proceeds by restoration and opposition, one obtains an estate and 24 *zuzim* equal 10 things [we omit the derivation of the last equality from the previous one]. This is the fifth type of the six types of restoration. Do as is written there, by taking half the number of roots, 5, squaring it, and it becomes 25. Subtract from it the number of *zuzim*, 24 and 1 is left. Take its root, which is also 1, and subtract it from the number of half the roots, 5. The remainder is four, and it is the one part of the ten. The other one is six. Also, if one adds the one to half the number of estates it becomes 6; it is one part of the ten and the other is 4.

SOURCES

I. Practical and Scholarly Arithmetic

1. Abraham ibn Ezra, *Sefer Hamispar*, translated by Roi Wagner from the Hebrew and German from Abraham ibn Esra. 1895. *Sefer ha-Mispar, Das Buch der Zahl*, edited and translated by Moritz Silberberg. Frankfurt am Main: J. Kauffmann, pp. 1–4, 12, 32–33, 45, 51–52, 56–58, 62–63, 70–71, 73–74,
2. Aaron ben Isaac, *Arithmetic*, translated by Naomi Aradi and Roi Wagner from the manuscript, Turin, Biblioteca Nazionale Universitaria A V 15 (IMHM: f 41584), fifteenth century, 18b–21a.
3. Immanuel ben Jacob Bonfils, *On Decimal Numbers and Fractions*, translated by Roi Wagner based on Solomon Gandz. 1936. “The Invention of the Decimal Fractions and the Application of the Exponential Calculus by Immanuel Bonfils of Tarascon (c. 1350),” *Isis* 25(1), 16–45, and Tony Lévy. 2003. “Immanuel Bonfils (XIVe s.): Fractions décimales, puissances de 10 et opérations arithmétiques,” *Centaurus* 45, 284–304, with the assistance of a somewhat deviant third manuscript edited by Naomi Aradi.
4. Jacob Canpanṭon, *Bar Noten Taʾam*, translated by Naomi Aradi and Roi Wagner from the manuscript London, British Library Or. 1053 (IMHM: f 5932), ff. 1b–65b (cat. Margo. 1012, 1), fifteenth century, pp. 34b–40b.
5. Elijah Mizraḥi, *Sefer Hamispar*, translated by Stela Segev from the edition in Stela Segev. 2010. “*The Book of the Number* by Elijah Mizraḥi: A Mathematical Textbook from the 15th Century (Hebrew).” Ph.D. thesis, Hebrew University of Jerusalem, and from the printed edition, Elijah Mizraḥi. 1534. *Sefer Hamispar* (Hebrew). Constantinople. Article I, part 1, pp. II-8, 43, 48, 56–57 65–68, 78, part 2, pp. 29a, 31a–b, Article III, part 1, chapter 1.
6. Levi ben Gershon, *Maʾase Ḥoshev*, translated by Shai Simonson from the Hebrew edition in Gerson Lange. 1909. *Sefer Maassei Chosheb—Die Praxis des Rechners, Ein hebraeisch arithmetisches Werk des Levi ben Gerschom aus dem Jahre 1321*. Frankfurt am Main: Louis Golde, after consulting the other available manuscripts. The problems at the end were already included in Shai Simonson. 2000b. “The Missing Problems of Gersonides: A Critical Edition,” *Historia Mathematica* 27, 243–302, 384–431.

II. Numerology, Combinatorics, and Number Theory

1. Abraham ibn Ezra, *Sefer Haʾeḥad*, translation revised by Roi Wagner from Tzvi Y. Langermann. 2001. “Studies in Medieval Hebrew Pythagoreanism: Translations and Notes to Nicomachus’ Arithmological

Texts," *Micrologus* 9, 219–236, based on the editions in Abraham ibn Esra. 1921. *Buch der Einheit*. Edited and translated by E. Müller. Berlin: Welt Verlag; Abraham ibn Ezra. 1856. "Sepher Haeḥad," *Jeschurun* 1, 3–10 (Hebrew); and Abraham ibn Ezra. 1867. *Sepher Ha-Echad*, edited and commented by S. Pinsker and M. A. Goldhart. Odessa: L. Nitzsche (Hebrew).

2. Abraham ibn Ezra. *Sefer Haʾolam*, based on Shlomo Sela. 2010. *Abraham ibn Ezra: The Book of the World*. Leiden: Brill.
3. Levi ben Gershon, *Maʾase Ḥoshev*, translated by Shai Simonson from the Hebrew edition in Gerson Lange. 1909. *Sefer Maassei Chosheb—Die Praxis des Rechners, Ein hebraeisch arithmetisches Werk des Levi ben Gerschom aus dem Jahre 1321*. Frankfurt am Main: Louis Golde.
4. Levi ben Gershon, *On Harmonic Numbers*, translated by Victor Katz from the Latin edition and French translation in Christian Meyer and Jean-François Wicker. 2000. "Musique et Mathématique au XIV^{e} Siècle: Le *De Numeris harmonicis* de Leo Hebraeus," *Archives internationales d'histoire des sciences* 50, 30–67.
5. Qalonymos ben Qalonymos, *Sefer Melakhim*, adapted and translated by Naomi Aradi and Roi Wagner from the Hebrew and French in Tony Lévy. 1996. "L'histoire des nombres amiables. Le témoignage des textes hébreux médiévaux," *Arabic Sciences and Philosophy* 6, 63–87.
6. Don Benveniste ben Lavi, *Encyclopedia*, adapted and translated by Naomi Aradi and Roi Wagner from the Hebrew and French in Tony Lévy. 1996. "L'histoire des nombres amiables. Le témoignage des textes hébreux médiévaux," *Arabic Sciences and Philosophy* 6, 63–87.
7. Aaron ben Isaac, *Arithmetic*, translated by Naomi Aradi and Roi Wagner from the manuscript, Turin, Biblioteca Nazionale Universitaria A V 15 (IMHM: f 41584), fifteenth century, 206b–208a.

III. Measurement and Practical Geometry

1. Abraham ibn Ezra (?), *Sefer Hamidot*, adapted with minor changes from the English translation in Tony Lévy and Charles Burnett. 2006. "'Sefer ha-Middot': A Mid-Twelfth-Century Text on Arithmetic and Geometry Attributed to Abraham ibn Ezra," *Aleph* 6, 57–238.
2. Abraham bar Ḥiyya, *Ḥibur Hameshiḥa Vehatishboret*, translated by Roi Wagner from the Hebrew-German edition, Abraham Bar Chija. 1913. *Chibbur ha-Meschicha weha-Tischboret: Lehrbuch der Geometrie*, edited and translated by M. Guttman. Berlin: Vereins Mekize Nirdamim.
3. Rabbi Shlomo Iṣhaqi (Rashi), *On the Measurements of the Tabernacle Court*, translated by David Garber and Roi Wagner from *Talmud Bavli*, *Eruvin* 23b.
4. Simon ben Ṣemaḥ, *Responsa on Solomon's Sea*, translated by David Garber and Roi Wagner from Simon Ben Ṣemaḥ Duran. 1998. *Sefer haTashbets* (Hebrew). Jerusalem: Shlomo Aumann Institute and Jerusalem Institute.
5. Levi ben Gershon, *Astronomy*. The translations from chapter four are taken from Bernard L. Goldstein. 1985. *The Astronomy of Levi ben Gerson (1288–1344)*. New York: Springer, pp. 41, 43–47. The translations from chapter 49 are taken from J. L. Mancha. 1998. "Heuristic Reasoning: Approximation Procedures in Levi ben Gerson's Astronomy," *Archive for History of Exact Sciences* 52, 13–50, pp. 15–16.

IV. Scholarly Geometry

1. Levi ben Gershon, *Commentary on Euclid's Elements*, translated by Victor Katz from the French in Tony Lévy. 1992. "Gersonide, Commentateur d'Euclide: Traduction annoteé de ses gloses sur les *Éléments"* in Gad Freudenthal (ed.), *Studies on Gersonides: A Fourteenth-Century Jewish Philosopher-Scientist*. Leiden: E. J. Brill, 83–147, pp. 100–102, 104–115, 146–147. Translation checked against the Hebrew original by Gad Freudenthal.
2. Levi ben Gershon, *Treatise on Geometry*, translated by Victor Katz from the French in Tony Lévy. 1992. "Gersonide, Commentateur d'Euclide: Traduction annoteé de ses gloses sur les *Éléments"* in Gad Freudenthal (ed.), *Studies on Gersonides: A Fourteenth-Century Jewish Philosopher-Scientist*. Leiden: E. J. Brill, 83–147, pp. 100–102, 104–115, 146–147. Translation checked against the Hebrew original by Gad Freudenthal.

3. Qalonymos ben Qalonymos, *On polyhedra*, translation adapted from Tzvi Langermann. 2014. "Hebrew Texts on the Regular Polyhedra," in Nathan Sidoli and Glen Van Brummelen (eds.), *From Alexandria, through Baghdad: Surveys and Studies in the Ancient Greek and Medieval Islamic Mathematical Sciences in Honor of J. L. Berggren.* New York: Springer, 411–469, pp. 440–441.
4. Immanuel ben Jacob Bonfils, *On Measurement of the Circle*, translation adapted from Tony Lévy. 2012. "Immanuel ben Jacob of Tarascon (Fourteenth Century) and Archimedean Geometry: An Alternative Proof for the Area of a Circle," *Aleph* 12, 135–158.
5. Solomon ben Isaac, *On the Hyperbola and Its Asymptote*, translated by Roi Wagner from the Hebrew and French in Tony Lévy. 1989a. "Le chapitre I,73 du Guide des Egarés et la tradition mathématique Hébreaïque au moyen age. Un commentaire inédit de Salomon b. Isaac," *Revue des études juives* 98(3–4), 307–336.
6. Abner of Burgos (Alfonso di Valladolid), *Sefer Meyasher ʿAqov*. Proposition 23 translated from the Hebrew by Tzvi Langermann in "Medieval Hebrew Texts on the Quadrature of the Lune," *Historia Mathematica* 23, 31–53; propositions 29–32 translated by Avinoam Baraness from the Hebrew edition in Gluskina, Gitta (ed). 1983. *Meyasher ʿAqob*. Moscow: Izdatelstvo Nauka and the manuscript London, British Library Add. 26984 (IMHM: f 5656; cat. Neub. 1002), fifteenth century.

V. Algebra

1. Quadratic word problems: a. Levi ben Gerson, Maʾase Ḥoshev, translated by Shai Simonson from the Hebrew edition in Gerson Lange. 1909. Sefer Maassei Chosheb—Die Praxis des Rechners, Ein hebraeisch arithmetisches Werk des Levi ben Gerschom aus dem Jahre 1321. Frankfurt am Main: Louis Golde. b. Anonymous arithmetic, translated by Naomi Aradi from the manuscript Bibliothèque de Genève, MS héb. 10/1 (IMHM f 2320), fols.2a–38a, fourteenth–fifteenth century. This translation also appears in Naomi Aradi. 2013. "An Unknown Medieval Hebrew Anonymous Treatise on Arithmetic," Aleph 13, 235–309. The particular texts are from fols. 25b, 26b, 29a. c. Elijah Mizraḥi, Sefer Hamispar, see I, 5. These problems are on pp. 92, 94, 96.
2. Simon Moṭoṭ, *Algebra*, translated by Roi Wagner from Naomi Aradi's Hebrew transcript of the manuscript Berlin Staatsbibliothek Or. Oct. 244/14 (IMHM: f 1996), ff. 113a–120a, fifteenth century with help from the French translation in Gustavo Sacerdote. 1893–1894. "Le livre de l'algèbre et le problème des asymptotes de Simon Moṭoṭ," *Revue des études juives* 27, 91–105; 28, 228–246; 29, 111–126.
3. Ibn al-Aḥdab, *The Epistle of the Number*, translated by Ilana Wartenberg in Ilana Wartenberg. 2007. "The *Epistle of the Number* by Isaac ben Solomon ibn al-Aḥdab (Sicily, 14th century): An Episode of Hebrew Algebra." Ph.D. thesis, Université de Paris 7, Tel Aviv University, pp. 443–459, 462–468, 539–560.

REFERENCES

Aradi, Naomi. 2013. "An Unknown Medieval Hebrew Anonymous Treatise on Arithmetic," *Aleph* 13, 235–309.

Archibald, Raymond C. 1915. *Euclid's Book on Division of Figures*. Cambridge: Cambridge University Press.

Baer, Yitzhak. 1961–1966. *A History of the Jews in Christian Spain*. Philadelphia: Jewish Publication Society of America.

Berggren, J. Lennart. 2005. "Mathematics in Medieval Islam," in Victor J. Katz (ed.), *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*. Princeton, NJ: Princeton University Press, pp. 515–675.

Clagett, Marshall. 1964–1984. *Archimedes in the Middle Ages*. Madison: University of Wisconsin Press and Philadelphia: American Philosophical Society.

Curtze, Maximilian. 1902. "Der 'Liber Embadorum' des Savasorda in der Übersetzung des Plato von Tivoli," *Abhandlungen zur Geschichte der Mathematischen Wissenschaften* 12, 1–183.

Dauben, Joseph. 2007. "Chinese Mathematics," in Victor J. Katz (ed.), *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*. Princeton, NJ: Princeton University Press, pp. 187–384.

David, F. N. 1998. *Games, Gods and Gambling: A History of Probability and Statistical Ideas*. New York: Dover.

Djebbar, Ahmed. 1992. "Le traitement des fractions dans la tradition mathématique arabe du Maghreb," in P. Benoit, K. Chemla, and J. Ritter (eds.), *Histoire de fractions, fractions d'histoire*. Basel: Birkhäuser, pp. 223–245.

Freudenthal, Gad. 1988. "Maimonides' *Guide for the Perplexed* and the Transmission of the Mathematical Tract 'On Two Asymptotic Lines' in the Arabic, Latin and Hebrew Medieval Traditions," *Vivarium* 26(2), 113–140.

———. (ed.), 1992. *Studies on Gersonides: A Fourteenth-Century Jewish Philosopher-Scientist*. Leiden: E. J. Brill.

———. 2005. "Two Notes on *Sefer Meyasher ʿAqov* by Alfonso, alias Abner of Burgos," in G. Freudenthal, *Science in the Medieval Hebrew and Arabic Traditions*. Farnham, UK: Ashgate/ Variorum Collected Studies, part IX.

Friedlein, Godofredi. 1873. *Procli Diadochi In primum Euclidis Elementorum librum commentarii*. Lipsiae: B. G. Teubneri.

Garber, David, and Boaz Tsaban. 1998. "Mathematical Appendix to the TASHBETS Responsa" (Hebrew), in Simon ben Ṣemaḥ Duran (ed.), 1998. *Sefer ha Tashbets* book (Hebrew). Jerusalem: Shlomo Aumann Institute and Jerusalem Institute, pp. 423–446. Available at http://u.cs.biu.ac.il/~tsaban/Pdf/tashbets.pdf.

———. n.d. "An Early Source for a Mathematical Method Appearing in the TASHBETS Responsa Book" (Hebrew). Available at http://u.cs.biu.ac.il/~tsaban/Pdf/rt.pdf.

Ginsburg, Jekuthiel, and David E. Smith. 1922. "Rabbi Ben Ezra and Combinations," *Mathematics Teacher* 15(6), 347–356.

Gluskina, Gitta (ed). 1983. *Meyasher Aqob*. Moscow: Izdatelstvo Nauka.

Hacker, Joseph. 2007. "Mizraḥi, Elijah," in M. Barenbaum and F. Skolnik (eds.), *Encyclopedia Judaica*, second edition. Gale Virtual reference library, vol. 14, 393–395.

Hacohen, Naftali Yaacov. 1967–1970. *Otzar Hagedolim Alufey Yaakov* (Hebrew). Haifa.

Harbili, Anissa. 2011. "Les Procédés d'Approximation dans les Ouvrages Mathématiques de l'Occident Musulman," *Llull* 34, 39–60.

Heath, T. L. 1956. *Euclid, The Thirteen Books of the Elements*, vol. I. Mineola, NY: Dover.

———. 1964. *Diophantus of Alexandria: A Study in the History of Greek Algebra*. Mineola, NY: Dover.

———. 1981. *A History of Greek Mathematics*, vol. I. Mineola, NY: Dover.

Hogendijk, J. P. 1993. "The Arabic Version of Euclid's *On Division*," in M. Folkerts and J. P. Hogendijk (eds.), *Vestigia Mathematica: Studies in Medieval and Early Modern Mathematics in Honour of H. L. L. Busard.* Amsterdam: Rodopi, pp. 143–162.

Høyrup, Jens. 2007. *Jacopo da Firenze's Tractatus algorismi and Early Italian Abbacus Culture*. Basel: Birkhäuser.

Ibn Ezra, Abraham. 1991. "Commentaries to Exodus," in *Torat Hayim.* Jerusalem: Mosad Harav Kook.

Imhausen, Annette. 2007. "Egyptian Mathematics," in Victor J. Katz (ed.), *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*. Princeton, NJ: Princeton University Press, pp. 7–56.

Kaplan, A. 1995. *Sefer Yetzirah, The Book of Creation*. Lanham, MD: Jason Aronson.

Knorr, Wilbur. 1989. *Textual Studies in Ancient and Medieval Geometry*. Boston: Birkhäuser.

Lange, Gerson. 1909. *Sefer Maassei Chosheb—Die Praxis des Rechners, Ein hebraeisch arithmetisches Werk des Levi ben Gerschom aus dem Jahre 1321*. Frankfurt am Main: Louis Golde.

Langermann, Tzvi. 1996. "Medieval Hebrew Texts on the Quadrature of the Lune," *Historia Mathematica* 23, 31–53.

———. 2001. "Abraham bar Ḥiyya Savasorda," in T. Hockey (ed.), *The Bibliographical Encyclopaedia of Astronomers*. New York: Springer, pp. 95–96.

———. 2014. "Hebrew Texts on the Regular Polyhedra," in Nathan Sidoli and Glen Van Brummelen (eds.), *From Alexandria, through Baghdad: Surveys and Studies in the Ancient Greek and Medieval Islamic Mathematical Sciences in Honor of J. L. Berggren*, New York: Springer, pp. 411–469.

Langermann, Tzvi, and Shai Simonson. 2000. "The Hebrew Mathematical Tradition," in Helaine Seline (ed.),*Mathematics across Cultures*. Dordrecht: Kluwer, pp. 167–188.

Levey, Martin. 1952. "The Encyclopedia of Abraham Savasorda: A Departure in Mathematical Methodology," *Isis* 43(3), 257–264.

———. 1954. "Abraham Savasorda and His Algorism: A Study in Early European Logistic." *Osiris* 11, 50–64.

———. 1966. *The Algebra of Abū Kāmil: Kitāb fi al-Jābr waʾl Muqābala in a Commentary by Mordecai Finzi*. Madison: University of Wisconsin Press.

———. 1989a. "Le chapitre I,73 du Guide des Egarés et la tradition mathématique Hébreaïque au moyen age. Un commentaire inédit de Salomon b. Isaac," *Revue des études juives* 98(3–4), 307–336.

———. 1989b. "L'étude des sections coniques dans la tradition medieval Hébraïque. Ses relations avec les traditions Arabe et Latine," *Revue d'histoire des sciences* 42(3), 193–239.

———. 1992. "Gersonide, Commentateur d'Euclide: Traduction annoteé de ses gloses sur les *Éléments*," in Gad Freudenthal (ed.), *Studies on Gersonides: A Fourteenth-Century Jewish Philosopher-Scientist*. Leiden: E. J. Brill, pp. 83–147.

———. 1996. "L'histoire des nombres amiables. Le témoignage des textes hébreux médiévaux," *Arabic Sciences and Philosophy* 6, 63–87.

———. 2001. "Les Débuts de la Literature Mathématique Hébraïque: La Géometrie d'Abraham bar Ḥiyya," *Micrologus* 9, 35–64.

———. 2002. "A Newly-Discovered Partial Hebrew Version of al-Khwarizmi's Algebra," *Aleph* 2, 225–234.

———. 2003. "L'algèbre arabe dans les textes hébraïques (I), Un Ouvrage Inédit D'Isaac ben Solomon al-Aḥdab (XIV siècle)," *Arabic Sciences and Philosophy* 13, 269–301.

———. 2007. "L'algèbre arabe dans les textes hébraïques (II). Dans l'Italie des XV^{e} et XVI^{e} siècles, sources arabes et sources vernaculaires," *Arabic Sciences and Philosophy* 17, 82–87.

———. 2008. "Abraham bar Ḥiyya," In H. Selin (ed.), *Encyclopaedia of the History of Science, Technology and Medicine in Non-Western Cultures*. New York: Springer, p. 5.

———. 2012. "Immanuel ben Jacob of Tarascon (Fourteenth Century) and Archimedean Geometry: An Alternative Proof for the Area of a Circle," *Aleph* 12, 135–158.

Lévy, Tony, and Charles Burnett. 2006. "'Sefer ha-Middot': A Mid-Twelfth-Century Text on Arithmetic and Geometry Attributed to Abraham ibn Ezra," *Aleph* 6, 57–238.

Maimonides, Moses. 1963. *The Guide for the Perplexed*. Chicago: University of Chicago Press.

Mancha, J. L. 1998. "Heuristic Reasoning: Approximation Procedures in Levi ben Gerson's Astronomy," *Archive for History of Exact Sciences* 52, 13–50.

Mazars, G., 1992. "Les fractions dans l'Inde ancienne de la civilisation de l'Indus à Mahavira (IXe siècle)," in P. Benoit, K. Chemla, and J. Ritter (eds.), *Histoire de fractions, fractions d'histoire*. Basel: Birkhäuser, pp. 209–218.

Millás Vallicrosa, Jose M. 1931. *Abraam bar Hiia, Libre de Geometria. Hibbur Hameixihà vehatixboret*. Barcelona: Alpha.

———. 1952. *La obra enciclopedica Yesode ha-Tebuna u-Migdal ha-Emuna de R. Abraham bar Ḥiyya Ha-Bargeloni*, edición critica con traducción, prólogo y notas. Madrid and Barcelona: Casa Provincial de Caridad de Barcelona.

Montelle, Clemency. 2014. "Hypsicles of Alexandria and the Fourteenth Book of Euclid's *Elements*." (in press)

Münster, Sebastian. 1546. *Sphaera mundi, autore Rabbi Abrahamo Hispano filio R. Haijae. Arithmetica secundum omnes species suas autore Rabbi Elija Orientali.* Basel: H. Petrum.

Ovadia, A. 1939. "Rabi Elija Mizraḥi," *Sinai* 5: 397–413, *Sinai* 6: 73–80.

Plofker, Kim. 2002. "Use and Transmission of Iterative Approximations in India and the Islamic World," in Y. Dold-Samplonius et al. (eds.), *From China to Paris: 2000 Years Transmission of Mathematical Ideas*. Stuttgart: Franz Steiner, pp. 167–186.

Rabinovitch, Nachum. 1970. "Rabbi Levi ben Gershon and the Origins of Mathematical Induction," *Archive for History of Exact Sciences* 6, 237–248.

Rangācārya, M. (ed.) 1912. *Mahāvīra's Ganitāsarasangraha*. Madras: Government Press.

Rashed, Roshdi. 2011–2014. *A History of Arabic Sciences and Mathematics*, edited by Nader El-Bizr. London and New York: Routledge and Beirut: Centre for Arab Unity Studies.

Rosenfeld, B. A. 1988. *A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space.* Translated from the Russian by Abe Shenitzer. New York: Springer.

Rubio, Mercedes. 2000. "The First Hebrew Encyclopedia of Science: Abraham bar Ḥiyya's *Yesodei ha-Tevunah u-Migdal ha-Emunah*," in S. Harvey (ed.), *The Medieval Hebrew Encyclopedias of Science and Philosophy*. Dordrecht, Boston, and London: Kluwer, pp. 140–153.

Sacerdote, Gustavo. 1893–1894. "Le livre de l'algèbre et le problème des asymptotes de Simon Moṭoṭ," *Revue des études juives* 27, 91–105; 28, 228–246; 29, 111–126.

Sadik, Shalom. 2012. "Abner of Burgos," in Edward N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*, Fall 2012 edition. Available at http://plato.stanford.edu/archives/fall2012/entries/abner-burgos.

Sefrin-Weis, H. (ed.). 2010. *Pappus of Alexandria: Book 4 of the Collection*. New York: Springer.

Segev, Stela. 2010. "*The Book of the Number* by Elijah Mizraḥi: A Mathematical Textbook from the 15th Century" (Hebrew). Ph.D. thesis, Hebrew University of Jerusalem.

Sela, Shlomo. 2003. *Abraham ibn Ezra and the Rise of Medieval Hebrew Science*. Leiden: Brill.

Sela, Shlomo, and Gad Freudenthal. 2006. "Abraham ibn Ezra's Scholarly Writings: A Chronological Listing," *Aleph* 6, 13–55.

Simonson, Shai. 2000a. "The Mathematics of Levi ben Gershon, the Ralbag," *Bekhol Derakhekha Daehu* (Bar-Ilan University) 10, 5–21.

———. 2000b. "The Missing Problems of Gersonides: A Critical Edition," *Historia Mathematica* 27, 243–302, 384–431.

Solon, Peter. 1970. "The *Six Wings* on Immanuel Bonfils and Michael Chrysokokkes," *Centaurus* 15(1), 1–20.

Steinschneider, Moritz. 1870. "Das Königsbuch des Kalonymos," *Jüdische Zeitschrift für Wissenschaft und Leben* 8: 118–122.

———. 1893–1901. *Mathematik bei den Juden*. Berlin, Leipzig, Frankfurt: Kaufmann; reprint. Hildesheim: G. Olms, 1964 and 2001.

———. 1906. "Mathematik bei den Juden,1551–1840," *Monatsschrift für die Geschichte und Wissenschaft des Judenthums* 2, 195–214.

Sude, Barbara Hooper. 1974. "Ibn al Haytham's Commentary on Euclid's *Elements* (*Sharh Musadarat Kitab Uqlidis Fi-Al Usul*), Books I–VI." Ph.D. thesis, Princeton University.

Suter, Heinrich. 1901. "Das Rechenbuch des Abû Zakarijâ al-Ḥassâr," *Bibliotheca Mathematica* 2, 12–40.

———. 1986. "Die Kreisquadratur des Ibn al-Haitam," in *Beiträge zur Geschichte der Mathematik und Astronomie im Islam*, Vol. 2. Frankfurt am Main: Institut für Geschichte der arabisch-islamischen Wissenschaften, pp. 76–90.

Toomer, G. J. 1984. *Ptolemy's* Almagest. New York: Springer.

———. 2008. "Nicomedes," in *Complete Dictionary of Scientific Biography*. Retrieved September 15, 2012 from Encyclopedia.com: http://www.encyclopedia.com/doc/1G2-2830903168.html.

Wagner, Roy. 2013. " Mordekhai Finzi's Translation of Maestro Dardi's Italian Algebra," in A. Fidora, H. Hames, and Y. Schwartz (eds.), *Latin-into-Hebrew: Texts in Contexts.* Leiden: Brill, pp. 195–221, 437–501.

Wartenberg, Ilana. 2007. "*The Epistle of the Number* by Isaac ben Solomon ibn al-Aḥdab (Sicily, 14th Century): An Episode of Hebrew Algebra." Ph.D. thesis, Université de Paris 7, Tel Aviv University,

———. 2008a. “*The Epistle of the Number*: An Episode of Algebra in Hebrew,” *Zutot: Perspective on Jewish Culture* 5, 95–101.

———. 2008b. “*Iggeret ha-Mispar*: by Isaac ben Solomon ibn al-Aḥdab (Sicily, 14th century) (Part I: The Author),” *Judaica* 64(1), 18–36.

———. 2008c. “*Iggeret ha-Mispar*: by Isaac ben Solomon ibn al-Aḥdab (Sicily, 14th century) (Part II: The Text),” *Judaica* 64(2/3), 149–161.

———. 2013. “*The Epistle of the Number*: The Diffusion of Arabic Mathematics in Medieval Europe,” in G. Federici-Vescovini and A. Hasnaoui (eds.), *Circolazione Dei Saperi Nel Mediterraneo: Filosofia E Scienze (Secoli IX–XVII): atti del VII Colloquio internazionale della Société internationale d'histoire des sciences et de la philosophie arabes et islamiques* (Florence, February 16–28 , 2006). Florence: Cadmo, pp. 101–110.

———. 2014. “The Naissance of the Medieval Hebrew Mathematical Language as Manifest in Ibn al-Aḥdab's Iggeret ha-Mispar,” in N. Vidro, I. Zwiep, and J. Olszowy-Schlanger (eds.), *A Universal Art: Hebrew Grammar across Disciplines and Faiths*. Leiden: Brill, pp. 117–131.

Wertheim, Gustav. 1896. *Die Arithmetik des Elia Misrachi. Ein Betrag zur Geschite der Mathematik, Zweite verbesserte Auflage*. Frankfurt am Main: Braunschweig, Friedich Vieweg und Sohn.

Woepcke, F. 1853. *Extrait du Fakhrī, traité d'algèbre par Abou Bekr Mohammed ben Alhaçan al-Karkhī précédé d'un mémoire sur l'algèbre indeterminée chez les Arabes*. Paris: L'imprimerie Impériale.

Youschkevitch, Adolf P. 1976. *Les mathématiques arabes: VIII*[e]*–XV*[e] *siècles*. Paris: Vrin.