# The Missing Problems of Gersonides – A Critical Edition – Part II

**By**

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## Abstracts

Gersonides' *Maaseh Hoshev,* (*The Art of Calculation*), is a major work known for its early use of rigorous combinatorial proofs and mathematical induction. There is a large section of problems at the end of the book, with the theme of proportions, which until now remained unpublished. I present a critical edition of this material. I also uncover a previously unknown second edition of *Maaseh Hoshev*. The material is appropriate for creative pedagogy and provides economic details of the author's culture and environs. A previous article presented the first fifteen problems and this article presents the rest.

*מעשה חושב* לרלב"ג הוא חיבור חשוב הידוע הודות לשימושו המוקדם בהוכחות קומבנטוריות מדויקות ובאינדוקציה מתימטית. בסוף הספר מופיע פרק גדול של שאלות שנושאן יחסים, שטרם פורסם. אני מציג הוצאה ביקורתית של פרק זה. החומר חושף מהדורה שנייה לא ידועה של *מעשה חושב*. החומר מתאים לפדגוגיה יצירתית ומספק פרטים כלכליים על אורח חייו וסביבותיו של המחבר. במאמר הקודם הוצגו חמש עשרה הבעיות הראשונות ומאמר זה מביא את היתר.

Le *Maase Hoshev* (*L'Art du Calcul*) de Gersonide est un ouvrage majeur connu pour son usage precoce de preuves combinatoires rigoureuses et de l'induction mathematique. La fin de l'ouvrage comporte une section importante consacree a une serie de trente problemes sur le theme des proportions qui etait restee inedite jusqu'a ce jour et don't je presente pour la premiere fois une edition critique. Je revele egalement l'existence d'une seconde edition de *Maase Hoshev*. Le materiau peut servir de base a des projets de pedagogie creative et offre des details sur la culture et l'environement de l'auteur. Cet article fait suite a une premiere etude qui presentait les quinze premiers problemes de *Maase Hoshev* et en presente les quinze autres.

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## I. Introduction

This paper is the second half of a critical edition with translation and commentary of a collection of heretofore unpublished problems, by the noted medieval scientist, philosopher and mathematician, Levi ben Gershon, a.k.a. Gersonides. The publication of these problems coincides with my discovery of a second edition of *Maaseh Hoshev*, the book which contains these problems. Please see part one of this critical edition, in the previous issue of this journal, for a detailed introduction, context and further analysis.

## II. Translation of Problems 16-21 from *Maaseh Hoshev* with Commentary

**(Ed. 1) 16. Problem:** We multiply one number by another and get the result. The sum of the two numbers is given. What are each of the numbers?

This is a famous ancient problem on quadratic equations, reappearing throughout history, and Levi gives the well known Babylonian presentation and solution. This is the only problem in the book that does not use proportions. The treatment here is very different than the Arabic algebra which preceded Levi. Levi considers only the simplest of quadratic equations, and does not pursue any broader categorization. He also does not present any geometric context

It asks, given M and N, to find x and y, such that $x+y = M$ and $xy = N$.
The solution given below is: $x = M/2 + \sqrt{((M/2)^2-N)}$ and $y = M/2 - \sqrt{((M/2)^2-N)}$.

Take the square of half the sum of the two numbers, and subtract the result from it. Take the square root of what remains, and add it to half the sum of the two numbers, to get the first number. If we subtract it from this half, you get the second number.

For example, the sum of two numbers is 13, and their product is 17. We know that the square of half of 13, is 42 and a quarter. Subtract 17, leaving 25 and a quarter. Extract the square root, to get 5 whole and one first, 29 seconds, 46, 34. Add this to 6 and a half, which is half of 13, to get the first number: 11 whole, 31 firsts, 29, 46, 34. The second number is: one whole, 28 firsts, 30, 13, 26. The product of one with the other is 17 to a very close approximation.

The numbers here are given in base 60. Levi, of course, uses spaces, not commas, to separate one base 60 digit from another.

It is impossible to find this number exactly, because 25 and a quarter does not have a true square root, as was explained. This is because the ratio of 25 and a quarter, to 25, equals the ratio of a hundred and one, to a hundred. But the ratio of a hundred and one, to a hundred, is not equal to the ratio of a square to a square. Since if this was the case, a hundred and one would be a square, because a hundred is a square. But if a hundred and one was a square, then its square root would be a whole number, and that is false.

Clearly Levi knew the all rational square roots are integers. See his Biblical commentary regarding the value of $\pi$ in [32].

If the problem was: we multiply a given number by a fixed part of itself; and we add the result to the product of this part with the remaining part of the given number; and the answer is given; what are each of the parts? Take the square of the whole number, and subtract from it, the sum, composed of the product of the number with a part of itself, and the product of this one part with the other part. Take the square root of what remains, and this is one part. What remains from the number is the fixed part.

> This asks, given M and N, to find x and y, such that $x+y = M$, and $Mx+xy = N$.
> The solution is: $y = \sqrt{(M^2-N)}$ and $x = M-y = M-\sqrt{(M^2-N)}$.

For example, the product of ten with a given part of itself, plus the product of this part with the second part, equals eighty. We want to know: what is the given part? The square of ten is a hundred. We subtract eighty from this to get twenty. We extract the square root, which is approximately 4 whole, 28, 19, 41, 21, and this is one part. What remains is 5 whole, 31, 40, 18, 39, which is the given part. If you multiply these parts by ten and by the leftover, you get eighty to a very close approximation.

If the problem was: we multiply a given number with a given part of itself; and we add the result to the square of the remaining part; and the answer is given; and we want to know: what are each of the parts? Subtract it all from the square of the whole number. Subtract what remains, from the square of half the number. Extract the root of what now remains, and add it to the half the number. This is the first part. What remains from the number is the second part.

> This asks, given M and N, to find x and y, such that $x+y = M$ and $Mx+y^2 = N$.
> The solution is: $y = M/2 + \sqrt{((M/2)^2-(M^2-N))}$ and $x = M-y = M/2-\sqrt{((M/2)^2-(M^2-N))}$.

For example, the product of ten with one part of itself, plus the square of the other part, is eighty. We want to know: what is each one of the parts? We subtract 80 from a hundred, and get twenty leftover. We subtract twenty from 25, and get five leftover. We extract the square root which is approximately 2 whole, 14, 9, 50, 40. We add this to five to get the first part, which is: 7 whole, 14, 9, 50, 40. The second part is: 2 whole, 45, 50, 9, 20.

> In standard Hebrew notation 59 is indistinguishable from 50 9 except for the space between the letter for 50 and the letter for 9. None of the mss. correctly include the space, likely due to careless and/or ignorant scribes.

The product of 10 with either one of these parts, plus the square of the other part, is approximately eighty. It is clear with a little investigation from our previous discussion, that whichever part is multiplied by ten, and added to the square of the other part, the result is one and the same. This is because the result is equal to the product of the first part with the second plus the squares of the two parts.

**(Ed. 1) 17. Problem:** We added one given number to certain fractions of a second given number and got a result equal to the sum of the second given number with different fractions of the first given number, and this result is known. What is each of the numbers?

> This problem is completely out of character with the rest of the section. It is almost ten times the length of the shorter problems. It spends a large percentage of its space, with detailed proofs which are all but unreadable. One proof uses no less than 27 variables causing Levi to exhaust all 22 letters in the Hebrew alphabet, and then use the special symbols for the 5 letters which look different when placed at the end of a word.

The method is to convert the fractions into unit fractions. After this, extract the numbers that meet the conditions, in the way that is explained in Theorem 52. This will be made clear from the proof therein. It is worth knowing that the all the explanations in theorems 47, 48 and 53 work for fractions even when they are converted to unit fractions. This will also be made clear from the proofs in the theorems.

> The references to theorems here are all accurate assuming the first edition numbering. The main theorem is 53 with the other theorems acting as lemmas, as we will see later.

After you complete the work of finding the numbers that meet the conditions, you will know the sum of the first given number with the given fractions of the second given number, and this is the number corresponding to the known result. You can extract the numbers corresponding to the two numbers, in the fashion described at the start of this section.

> The method "of finding the numbers that meet the conditions" is equivalent to finding x and y, such that X+(a/b)Y = Y+(c/d)X, where a/b and c/d are given fractions less than one. Levi's solution is: X = (d/c)((b/a)-1) and X = (b/a)((d/c)-1). The method is used below to find U and V, given M such that U+(a/b)V = V+(c/d)U = M. This is done by setting up the proportions X/U = Y/V = (X+(a/b)Y)/M = (Y+(c/d)X)/M.

For example, the first given number plus 2 sevenths and a ninth of the second given number is twenty. If you add the second given number plus 2 fifths of the first given number, you also get twenty. We want to know: what is each of the numbers?

We convert 2 sevenths plus a ninth to a unit fraction. This is one part, of two and 13 of 25 parts of one. We also convert 2 fifths to a unit fraction. This is one part, of two and a half. After this, we extract two numbers that meet these conditions, according to the explanation in Theorem 52. Accordingly, multiply one and 13 of 25 parts of one, by 2 whole and a half, to get 3 and 4 fifths, which is the first number. Multiply one and a half, by 2 and 13 parts of 25, to get 3 whole and 39 of 50 parts, which is the second number. The first number plus 2 sevenths and a ninth of the second number, equals 5 whole and 3 tenths, and so does the second number plus 2 fifths of the first number. Since you know that the number corresponding to 5 whole and 3 tenths is twenty, then using this ratio, extract the numbers corresponding to the first and second numbers, and these are what were requested. Multiply the first number by twenty and divide by 5 whole and 3 tenths, to get that the first given number equals 14 whole and 18 parts of 53. In the same fashion, the second given number equals 14 whole and 14 of 53 parts of one. And use this as a model.

This is right, because the ratio of the first number to the first given number, equals the ratio of the result to twenty, which equals the ratio of the second number to the second given number.

| First | Result | Second |
|---|---|---|
| 3 whole and 4 fifths | 5 whole and 3 tenths | 3 whole and 39 parts of 50 |
| Given First | Result | Given Second |
| 14 whole and 18 parts of 53 | Twenty | 14 whole and 14 parts of 53 |

Similarly, the ratio of 2 fifths of the first number, to 2 fifths of the first given number, equals the ratio of the first number to the first given number. Thus, the ratio of 2 sevenths and a ninth of the second number, to 2 sevenths and a ninth of the second given number, equals the ratio of the second number to the second given number, which equals the ratio of the first number to the first given number. By adding the two corresponding numbers, we also get that the ratio of their sum to the sum of their corresponding pairs, equals the ratio of the result to twenty. Accordingly, the ratio of the sum of the second number and 2 fifths of the first number, to the sum of the second given number and 2 fifths of the first given number, equals the ratio of the result, which is 5 whole and 3 tenths, to twenty. By exchanging them, the ratio of the sum of the second number and 2 fifths of the first number, to the result, equals the ratio of the sum of the second given number and 2 fifths of the first given number, to twenty. But the first of these four equals the second, so the third equals the fourth. Hence, the second given number plus 2 fifths of the first given number is twenty.

If the problem was to add multiples of the second number to the first number, and multiples of the first number to the second, so that the result is the same, then take one away from the multiples of the first number, and what remains is the second number. Similarly, take one away from the multiples of the second number, and what remains is the first number. You will find that these numbers meet the required conditions.

> Here Levi discusses a simpler method, which because he does not use negative numbers, works only when the fractions are greater than one. By "multiple", Levi means a number greater than one. The method is to find X and Y, such that X+(a/b)Y = Y+(c/d)X, where a/b and c/d are fractions greater than one. The solution is X = (a/b)-1 and Y = (c/d)-1, which is justified in the example that follows.

For example, the first number plus 2 times the second plus half the second, equals the second number plus 3 times the first plus a quarter of the first. Set the first at one and a half; and the second at two and a quarter. Accordingly, the first plus 2 times the second plus half the second, equals the sum of the first and the second, and the product of one and a half, which is the first, with two and a quarter, which is the second. Similarly, the second plus three times the first and a quarter of the first, equals the sum of the second and the first, and the product of the second, which is two and a quarter, with one and half, which is the first. Thus it is clear, that the result is one and the same, and it equals 7 whole and an eighth. If the problem stated that the first number is twenty, then you could also figure out the second number. Toward this end, the ratio of the first number to twenty, equals the ratio of the second, which is 2 and a quarter, to the unknown. Accordingly, the second number is thirty.

In theorems 48 and 53, this matter is explained for fractions and unit fractions with the very same proof, since we convert the fraction to a unit fraction. This is the same for Theorem 47, where the first number, after converting the fraction into a unit fraction, was two. But if it had been less than two, then the method of finding these numbers is not explained there.

As before, the theorem references are accurate and refer to the numbering in the first edition.

I claim that the whole matter is turned backwards, when the first number is greater than two. Then the first of these numbers is the third with the excess of the second over the first, minus the product of the subtraction of the first from two, with the product of the second and third. The second and third are extracted in the way explained there.

Levi is implicitly referring here to a problem discussed in detail in Theorem 53 and also in Problem 21. He does not explicitly state the problem or give any examples until he is finished with a long proof explaining why his new solution is correct in the case where “the first number is greater than two”. The original problem is to find x, y and z, given $a/b > c/d > e/f$, such that $X+(a/b)(Y+Z) = Y+(c/d)(X+Z) = Z+(e/f)(X+Y)$. His solution is $x = C+(B-A)+(A-2)BC$, $Y = X+2(B-A)(C-1)$, and $Z = Y+2(A-1)(C-B)$, where $A = b/a$, $B = d/c$, and $C = f/e$. The new solution is equivalent, but is necessary to avoid negative numbers in the case that the first number, a/b, is greater than two, that is, $A \leq 2$. By “backwards” Levi means that $x = C+(B-A)-(2-A)BC$, rather than $C+(B-A)+(A-2)BC$.

For example, after converting the fractions to a unit fraction, let the number which is less than two from which we take one part, be A.

That is, if we start with $a/b > 2$, and convert to a unit fraction to get $1/(b/a)$, then b/a is the number less than 2, that Levi calls A.

Let D be what remains after subtracting it from 2. Let B be the second number, and let E be its excess over A. Let C be the third number, and let G be its excess over B. Let H be the number preceding B, and I be the number preceding C, and L be the number preceding A. L is the excess of A over one.

That is: $D = 2-A$, $E = B-A$, $G = C-B$, $H = B-1$, $I = C-1$ and $L = A-1$.

Let K + L be equal to A, and accordingly K is one. We subtract the product DBC from the numbers C + E, and let what remains be M, which is the first number. Add M to twice the product of E with I, and let the result be N, which is the second number. Add N to twice the product of L with G, and let the result be O, which is the third number. We will explain why these numbers are what were requested.

That is: First = M = C + (B-A) – (2-A)BC
Second = N = M + 2(B-A)(C-1)
Third = O = N + 2(A-1)(C-B)

The proof, is that the first number, M, equals the number C + E minus the number DBC. The second number, N, equals the number C + E minus the product DBC plus twice the product of E with I. The third number, O, equals the number C + E minus the product DBC plus twice the product of E with I and twice the product of L with G. We claim that one of A parts of the number N + O, equals twice the product of H with I.

Throughout this proof and the rest of the problem, Levi juxtaposes variables next to each other, sometimes meaning sum and sometimes product. When he means product, he usually inserts the word *murkav* (composition) in front of the variables, and when he means sum he usually inserts the word *misparei* (numbers). I translate *murkav AB* as "the product AB", and *misparei AB*, as "the number A+B". If he does not use the appropriate leading word, then I use A+B or AB as the context demands.

The proof is, that half of the number N+O, is equal to the number C+E plus twice the product of E with I, plus L times G, minus the product DBC. Let's adjoin the product of L with H+C. Hence, half of the number N+O, is equal to the number C+E plus twice the product of E with I, plus L times G, plus the product of L with H+C, minus the product DBC and the product of L with H+C. But E times I, plus L times G, equals H times the sum of the numbers E and G; because we have three numbers L, H and I; the excess of H over L is E, and the excess of I over H is G. Now the number E+G+L equals the number I. Hence, the product of H with the number E+G, plus H times L, equals H times I. We subtract L times H, from L times H+C, and what remains equals L times C, plus the number C+E, plus E times I, minus the product DBC and the product of L with H+C. I claim that this all equals the product LHI.

The proof is, that we add E times I, to E, which equals E times C. Add C to the result, which then equals K+E, times C. Add L times C, to the result, which then equals E+K+L, times C. This equals B times C. This equals D+L, which is one, times the product BC. We subtract the product DBC, and what remains is the product LBC.

Now, the product LHI, subtracted from the product LBC, is the product LHC. This is because the product LH times I, plus the product LH, equals the product LHC; the product LHC equals H times LC; and when we add in the product LC, the result equals B times L+C.

There are two scribal errors in this last paragraph. In the first line, "the product LHC" should be "the product of L with H+C", and in the last line, "B times L+C" should be "B times the product LC". The first is the result of a missing letter meaning "with", and the second is the result of a missing word "product". The first error is repeated again in the next line, where "the product LHC" should be "the product of L with H+C".

Hence, when the product LHC is subtracted from the product LBC, which will always afford the subtraction, what remains is the product LHI.

Hence, half of the number N+O, is equal to the product KHI, which is H times I, plus the product LHI.  This equals K+L, times the product HI.  But K + L is A.  Hence, half of N+O, equals the product AHI.  Hence, H times I, is one of A parts of half of the number N+O.  Accordingly, twice the product of H with I, is one of A parts of the sum of the numbers N and O.  Let this result be P.

> This proof is hard to follow, especially in the Hebrew.  Besides, the standard difficulty of reading an algebraic idea presented in words, there is the additional burden of lookalike letters, and Levi's cumbersome notation.  Levi uses letters as variables and numbers.  To distinguish a normal letter from one used as a number or variable, he puts a small slash above the letter, like an apostrophe.  He uses adjacent letters as either or a number in the standard Hebrew form, or the sum or product of the variables represented by those letters.  He carefully says "the product ABC"  and "the numbers ABC", to distinguish product from sum, but he is not always consistent.  Furthermore, some combination of variables make words like "this" (GE) and "all" (EKL) which show up commonly as words in the proof.  What is worse, is that when the Hebrew letters corresponding to B, E and L, are adjoined to the beginning of a word, they mean, "with", "the" and "to" respectively.  Therefore, the presence or absence of the slashes above the letters, and the spacing is crucial for the correct reading, yet the scribes tend to be sloppy about the slashes and spacing, sometimes just putting a big slash over the whole word.  It is hard, for example, to distinguish "the product ABC" from "the product of A with C".
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> From the many minor errors and the noticeable lack of corrections in the margins, one gets the sense that very few people understood Levi's idea, certainly not the scribes or subsequent owners.  It is not surprising that the problem is omitted in its entirety from the second edition.

Furthermore, half of the number M+O, when we adjoin L times B, equals the number E+C, plus E times I, plus L times G, plus L times B, minus the product DBC and L times B.  I claim that it all equals the product LBI.  This is because E times I, plus E, equals E times C.  When we add E times C, to C, we get E plus K, times C.  Now, L times B, plus L times G, equals L times C.  Since E+K+L equals B, the total is L+D times the product BC, since L+D is one.  After subtracting the product DBC, what remains is the product LBC.  After subtracting the product LB, what remains is I times the product LB.  Hence, half the numbers M and O, equals the product BLI.  Hence, L times I, is one of B parts of half the sum of the numbers M and O.  Accordingly, twice the product of L with I, is one of B parts of the sum of the numbers M and O.  Let this result be Q.

> The last paragraph has a number of scribal errors in all six of the relevant extant mss. all of which are listed in the Hebrew part of this critical edition.  The reconstruction above is consistent with the content and the later references in each ms.  For example, all the extant mss. have Q in the last sentence above as  O.  However, the variable O has already been used for a different purpose, and if we use O instead of Q, then Q never gets defined, because the next new variable to be defined is R.  Yet Q is referenced many times later and Levi chooses his new variables in alphabetical order.  Hence, I believe that the reconstruction above is the original text of Levi, and that the mss. present it incorrectly.  Of course, the critical apparatus in the Hebrew, records all the variations.

Furthermore, half the number M+N, when we adjoin L times C, equals the number E+C, plus E times I, minus the product DBC and L times C.

Levi does not explicitly repeat the inclusion of L times C in the sum here, as he does in the two previous similar arguments with L times B, and L times H+C, respectively. This is either a choice in style or a careless error. However, he clearly intends for L times C to be included in the sum.

I claim that this all equals the product LHC. This is because, E times I, plus the number E+C, equals E+K, times C, as we explained before. When we add L times C, the result is E+K+L, times C, which equals B times the product LC. When we subtract off the product LC, what remains is H times the product LC. This equals the product LHC, which is C times the product LH. Hence, L times H, is one of C parts of half of the number M+N. Accordingly, twice the product of L with H, is one of C parts of the sum of the numbers M and N. Let this result be R.

I claim that the number M+P, equals the number N+Q, which equals the number O+R. This is because the number M+P, equals M plus twice the product of H with I. Furthermore, N+Q, equals M plus twice the product of E with I, plus twice the product of L with I. Hence, the number N+Q, also equals M plus twice the product of H with I. Furthermore, O+R, equals M plus twice the product of E with I, plus twice the product of G with L, plus twice the product of H with L. But twice the product of G with L, plus twice the product of H with L, equals twice the product of G+H, with L; and this equals twice the product of I with L. When we add in twice the product of I with E, the result is twice the product of I, with L+E. But L+E equals H, so the result is twice the product of I with H. When we add in M, the result is also equal to M plus twice the product of H with I. This is what we wanted.

A modern reconstruction of the whole proof is shown below, to further assist the reader.

Levi is trying to find X, Y and Z satisfying the equation below, where $a/b > c/d > e/f$ are given fractions.

$$X + (a/b)(Y+Z) = Y + (c/d)(X+Z) = Z + (e/f)(X+Y)$$

To this end, he defines:

| | | |
|---|---|---|
| $A = b/a$ | $D = 2 - A$ | $H = B - 1$ |
| $B = d/c$ | $E = B - A$ | $I = C - 1$ |
| $C = f/e$ | $G = C - B$ | $L = A - 1$ |

He claims that X = M, Y = N and Z = O satisfy the equation above, where:
$M = C + E - BCD$,
$N = M + 2EI$, and
$O = N + 2LG$.

Lemma 1: $EI + LG = H(E+G)$
Proof: By Theorem 45 in the first edition (Levi's reference is not explicit),
$I(H-L) + L(I-H) = H(I-L)$, where $I > H > L$.
Since $H-L = E$ and $I-H = G$, the lemma follows.

Lemma 2: $H(E+G) + HL = HI$
Proof: $E+G+L = I$.

Lemma 3: $LC + C + E + EI - DBC - L(H+C) = LHI$

Proof: Levi just adds up the left side step by step keeping a running total.

EI+E = EC
EC + C = C(E+1)
C(E+1) + LC = C(E+1+L) = BC = (D+L)BC
(D+L)BC – DBC = LBC
LBC – L(H+C) = LHI, since LH(I) + LH = LHC = H(LC), and H(LC) + LC = B(LC)

Lemma 4: (N+O)/A = 2(HI)
Proof: (N+O)/2 = C + E + 2EI + LG – DBC
= C + E + 2EI + LG + L(H+C) – DBC – L(H+C)
By Lemma 1,
= C + E + EI + H(E+G) + L(H+C) – DBC – L(H+C)
= C + E + EI + LC + LH + H(E+G) – DBC – L(H+C)
= LC + C + E + EI – DBC – L(H+C) + H(E+G) + HL
By Lemmas 2 and 3,
= LHI + HI = (L+1)HI = AHI = HI(A).
Hence, (N+O)/A = 2HI.

He then proves in a similar way:

Lemma 5: (M+O)/B = 2LI, and
Lemma 6: (M+N)/C = 2LH

With these preliminaries out of the way, Levi finishes his proof.

Theorem 1: M + (N+O)/A = N + (M+O)/B = O + (M+N)/C
Proof: By Lemma 4, (N+O)/A = 2HI, so M + (N+O)/A = M + 2HI. But N = M + 2EI and by Lemma 5, (M+O)/B = 2LI. Hence N + (M+O)/B = M + 2EI + 2LI = M + 2HI.
Finally, O = N + 2LG = M + 2EI + 2LG, and by Lemma 6, (M+N)/C = 2LH.
Hence O + (M+N)/C = M + 2EI + 2LG + 2LH = M + 2EI + 2LI = M + 2HI.

I will now give you examples for all possible kinds of fractions, so that you will become wise and understand.

In the examples that follow, when A > 2, Levi adds (A-2)BC, else if A < 2, he subtracts (2-A)BC. This is as he promised at the start of the whole discussion, in order to avoid negative numbers.

We want to find three numbers, such that one number plus 2 sevenths of the rest, equals another plus 2 fifths of the rest, and equals the other plus 3 parts of 11 of the rest. We convert 2 sevenths to a unit fraction, and get one part of 3 and a half. We convert 2 fifths to a unit fraction, and get one part of 2 and a half. We convert 3 parts of 11 to a unit fraction, and get one part of 3 and 2 thirds. We add the large number to the excess of the middle number over the small number, and we add on the product of the middle number with the large number with the excess of the small number over two. Thus the first number is 11 whole and one of 12 parts of one. We add on twice the product of the excess of the middle over the small, with one less than the large, to get 16 whole and 11 of 12 parts of one, which is the third number.

This is an error. The third number should be 16 5/12, and second number, which is not mentioned, should be 16 11/12. It seems that the phrase “which is the second number. We add on twice the product of the excess of the large over the middle, with one less than the small, to get 16 5/12" was lost in between the words "one," and "which" in the last line.

The first with 2 fifths of the others, which is one of 2 and a half parts, equals the second with 2 sevenths of the others, and equals the third with 3 of 11 parts of the others.

In example two, we want to find three numbers, such that one number with half of the rest, equals another with 2 fifths of the rest, and equals the other with 3 of 11 parts of the rest. When we convert to unit fractions, the numbers of which we have single parts are: two, two and a half, and 3 and 2 thirds. We make the first number equal to the sum of the large number and the excess of the middle over the small. Accordingly, the first number is 4 whole and one sixth. The second number, as before, is 6 whole and 5 sixths. The third number is 8 whole.

This is an error. The third number should be 9 1/6.

The first with half of the rest, equals the second with 2 fifths of the rest, and equals the third with 3 of 11 parts of the rest.

In example three, we want to find three numbers, such that the one with 3 fifths of the rest, equals another with 4 of 11 parts of the rest, and equals the other with 2 sevenths of the rest. When we convert these to unit fractions, the smallest of these numbers of which we have single parts, is one and 2 thirds; the middle is 2 and 3 fourths; and the large is 3 and a half. The first number, by the previous method, is one and 9 parts of 24. The second number is 8 and 2 parts of 24 and the third number is 9 and 2 parts of 24.

This is an error. The second number should be 6 19/24. The third number should be 7 19/24. The errors in the previous two examples are the result of ignorant and/or careless scribes. However, in this example the second error is correctly one greater than the first, so it is less likely.

Here is another easy way of finding numbers where the first plus one part or parts of the rest, equals the other plus one part or parts of the rest. For example, the first with A of D parts of the rest, equals the second with B of E parts of the rest, equals the third with one of G parts of the rest, and equals the fourth with C of H parts of the rest. We let A+I equal D, B+K equal E, L be one less than G, and C+M equal H. Multiply D times the product KLM, let the result be N, and this will be the sum of three of the numbers, without the first. Also, multiply E times the product ILM, let the result be O, and this will be the sum of three of the numbers without the second. Also, multiply G times the product IKM, let the result be P, and it will be the sum of three of the numbers without the third. Also, multiply H by the product IKL, let the result be Q, and this will be the sum of three of the numbers without the fourth.

This is a completely new method and Levi redefines all the variables starting with A. Although Levi starts the second sentence of the previous paragraph with the words "For example", he is actually describing the method in a completely general way, and gives specific examples later.

Levi computes the four unknown variables, First, Second, Third and Fourth by first assigning particular values to Second+Third+Fourth, First+Third+Fourth, First+Second+Fourth and

First+Second+Third, and then explaining how to derive the four variables from these sums assuming the sums are correct. His method is to calculate the differences between the largest sum and each of the others. Assuming the largest sum is First+Third+Fourth, this gives: First-Second, Third-Second, and Fourth-Second. Add these three differences together, subtract from First+Second+Third, and divide by three to get Second. The variables First, Third and Fourth can be gotten by substituting back into First-Second, Third-Second, and Fourth-Second.

After you know the sum of every three numbers out of the four, you can extract each one of the numbers in its proper place. The nicest way to do this, is to take the smallest of the numbers N, O, P and Q, and then the smallest of the rest, and so on until you reach the largest. Let's say the smallest was P, the next smallest N, the next smallest Q, and accordingly the largest O. Let R be the excess of O over Q, S be the excess of O over N, and T be the excess of O over P. Since O is the sum of the first, third and fourth, and Q is the sum of the first, second and third; then when we subtract off the first and third which they have in common, what remains is the excess of the fourth over the second, which is the excess of O over Q. Accordingly, the excess of the fourth over the second is R. Also, since O is the sum of the first, third and fourth, and N is the sum of the second, third and fourth; then when we subtract off the third and fourth which they have in common, what remains is the excess of the first over the second, which is S. Similarly, the excess of the third over the second is T, since the excess of O over P is T.

Consequently, the second number is the smallest. Take one of the sums that includes the second number, and let this be P in our example. Since the number P is the sum of the first, second and fourth, it is clear that the excess of P over three times the second, equals the sum of the excess of the first over the second, and the excess of the fourth over the second. This is the number R+S. Accordingly, subtract the number R+S from the number P, and divide what remains by the number of numbers that comprise the sum P, that is by three. The result from this division is the second number, and let this be U. Accordingly, the first number is the sum of U and S, the third number is the sum of the numbers U and T, and the fourth number is the sum of the numbers U and R.

Levi's method is a general one for finding n unknowns from n equations, where each of the equations gives the sum of a unique subset of n-1 of the unknowns.

It seems that Levi did not have a general method like Gaussian elimination for solving general systems of linear equations, or else he would have mentioned it, especially due to its recursive nature, a technique he is famous for pioneering. It is interesting that a general method identical to Gaussian elimination was known in China hundreds of years earlier.

Now Levi explains why his computation of Second+Third+Fourth, First+Third+Fourth, First+Second+Fourth and First+Second+Third, was correct in the first place, and why the resulting values for First, Second, Third and Fourth solve the problem.

Now that we have found all these numbers, we will explain why these numbers are the ones satisfying the requested conditions. To this end, we will explain that what remains from these numbers is always the product IKLM.

The phrase “what remains from these numbers”, means what remains from the sum First+Second+Third+Fourth.

That is, First+Second+Third+Fourth – (First + A/D(Second+Third+Fourth)) = IKLM,
First+Second+Third+Fourth – (Second + B/E(First+Third+Fourth)) = IKLM,
First+Second+Third+Fourth – (Third + 1/G(First+Second+Fourth)) = IKLM,
First+Second+Third+Fourth – (Fourth + C/H(First+Second+Third)) = IKLM.

The proof, is that when the first is added to A of D parts of the rest, what remains is I times the product KLM. This is because, the rest equals N, which is D times the product KLM. So there are as many of the product KLM in the number N, as there are ones in the number D. Now the product KLM is one of D parts of the number N. Accordingly, there are as many of the product KLM, in A of D parts of the number N, as there are ones in the number A. Thus there are as many of the product KLM in what remains of these three numbers, as there are ones in the number I, since the number I+A equals D. This equals the product IKLM.

Similarly, when the second is added to B of E parts of the rest, what remains equals K times the product ILM, which equals the product IKLM. Similarly, when the third is added to one of G parts of the rest, what remains equals L times the product IKM; this is because L is one less than G; this equals the product IKLM. Now since what remains from these four numbers is always equal to the product IKLM, then what you take from them are certainly one and the same. Because after subtracting equals from equals, the results are equal. This is what we wanted to explain.

The reader should note that the original problem is an underdetermined set of equations, and that Levi effectively adds a constraint. Namely, that the sum of all the variables equals each of the other equations plus IKLM, the product of the differences of the numerator and denominator in each given fraction. This extra constraint determines a unique solution.

Levi now claims that if S+R > P then the original problem has no positive solutions, but he proves only that if S+R > P then the determined set of equations has no positive solution. However, the original problem has a solution in positive integers, if and only if the unique solution determined by the original problem with the extra constraint is positive. This is true because the line representing the general solution always goes through the origin, and the extra constraint is equivalent to setting an undetermined variable to a positive value. Hence, Levi’s claim is correct, but his proof needs more detail, and he seems unaware of this need.

I now claim that if the sum of S and R, in our example, is greater than or equal to the number P, then the problem is certainly a fraud. That is, there is no way to find numbers that meet the conditions.

Levi, as usual, only considers positive solutions. If no positive solutions are possible, then he calls the problem a fraud.

In the upcoming proof, Levi introduces variables past the 22 letters in the Hebrew alphabet, so he uses letters that appear differently at the end of a word, than they do normally in the middle of a word. There are five such letters and he uses them all. This allows him 27 variables, so I use the Greek letter Ω when I run out of English letters.

Because of this complexity of notation, and because the scribes did not understand the proof they were copying, there are no mss. which correctly represent every variable in the way that I believe they were originally written. Not only could a scribe forget a slash, or omit a space, but he could write a specific letter either in its ending form, or in its normal form. This ambiguity confused the scribe who did not likely appreciate the serious error of substituting an end form for a normal form even if that letter happens to appear at the end of a list of variables. Nor did the scribe appreciate the similar error of substituting a normal form for an end form, even that letter happens to appear in the middle of a list of variables. The reconstruction below is an accurate composite based on the extant mss., the consistency of later references, the content of the proof, and Levi's style. Of course, the critical apparatus in the Hebrew, records all the variations.

The proof is, that if it were possible, let these numbers be V, W, X, Y. Then V, plus A of D parts of the number W+X+Y, equals W plus B of E parts of the number V+X+Y, or X plus one of G parts of V+W+Y, or Y plus C of H parts of the number V+W+X. Accordingly, what remains from these four numbers is one and the same, and let this be Z, by way of example.

We will explain that the ratio of Z to the sum of the numbers W, X and Y, equals the ratio of the numbers I to D. This is because when we take A of D parts of the number W+X+Y, what remains from them all, Z, is equal to I of D parts of the number W+X+Y. This is because the number A+I equals D. Similarly, the ratio of V+X+Y to Z, equals the ratio of E to K. Similarly, the ratio of V+W+Y to Z, equals the ratio of G to L. Similarly, the ratio of W+X+Y to Z, equals the ratio of H to M.

Now divide the number Z by as many ones as there are in the product IKLM, and let one part of the product IKLM, times Z, be Ω. Since the ratio of Z to the number V+W+Y, equals the ratio of L to G, and there are as many Ω's in Z as there are ones in L times the product IKM, then there are as many Ω's in the number V+W+Y as there are ones in G times the product IKM. Accordingly, there are as many Ω's in V+W+Y as there are ones in P. Similarly, there are as many Ω's in W+X+Y as there are ones in N; there are as many Ω's in V+X+Y as there are ones in O; and there are as many Ω's in V+W+X as there are ones in Q.

Accordingly, it is clear by an explanation similar to the previous one, that there are as many Ω's in the excess of the first over the second, as there are ones in S; and there are as many Ω's in the excess of the fourth over the second, as there are ones in R. Accordingly, it must be that there are as many Ω's in the excess of W+Y over two times V, as there are ones in S+R.

Levi is now referring to V,W, X and Y, as second, first, third and fourth respectively. This is inconsistent with his original assumption that V is first and W second , but does not otherwise invalidate the proof.

Now let F be the number of Ω's in V. We already know that there are as many Ω's in the number V+W+Y, as there are ones in the number S+R and Ω's in three times the number V. Hence, there are as many Ω's in the number V+W+Y, as there are ones in the number S+R plus three times the number F. But there are as many Ω's in the number

V+W+Y, as there are ones in the number P.  Hence, the number P equals the number S+R plus three times the number of ones in the number F.  But we already assumed that the number S+R was greater than or equal to the number P.  This is false;  that is, a part of something being equal or greater to the whole.  Consequently, the question is a fraud. This is what we wanted to explain.

> Levi finally uses the variable F, which completely exhausts his supply of Hebrew letters.  Levi has already used  the 21 other Hebrew letters plus the five extra "final letter" forms.  Note that the 6th Hebrew letter means "and", when it is adjoined to the beginning of a word.  Hence, Levi avoids using it until he has no option.

Now I will give you an example of this.

> In the example that follows, Levi uses "A" through "G" as shorthand for "first" through "seventh".

We want to find seven numbers, such that the first plus 2 sevenths of the rest, equals the second plus a third of the rest, equals the third plus 2 ninths of the rest, equals the fourth plus 3 eighths of the rest, equals the fifth plus a sixth of the rest, equals the sixth plus a fourth of the rest, and equals the seventh plus 2 of 11 parts of the rest. According to the previous explanation, multiply 7 by the product 2*7*5*5*3*9, as you will see in the following way.  The result is 66 thousand and 150, which is the sum of the second, third, fourth, fifth, sixth and seventh.  Multiply 3 by the product 5*7*5*5*3*9, and the result is 70 thousand and 875, which is the sum of A, C, D, E, F and G.  Multiply 9 by the product 5*2*5*5*3*9, to get 60 thousand and 750, which is the sum of A, B, D, E, F and G.  Multiply 8 by the product 5*2*7*5*3*9, to get 75 thousand and 600, which is the sum of A, B, C, E, F and G.   Multiply 6 by the product 5*2*7*5*3*9, to get 56 thousand and 700, which is the sum of A, B, C, D, F and G.  Multiply 4 by the product 5*2*7*5*5*9, to get 63 thousand, which is the sum of the first, second, third, fourth, fifth, sixth and seventh.  Multiply 11 by the product 5*2*7*5*5*3, to get 77 thousand and 750, which is the sum of A, B, C, D, E and F.

> This example has multiple errors in it, and if done correctly, results in the case that Levi just showed has no positive solutions!  The main error is that 77,750 should be 57,750.  This error propagates onward until it results in a wrong solution at the end.  Hence the error is almost surely Levi's.  There are other errors that occur, even assuming the 77,750 to be correct, which I comment on below where they appear.
>
> A minor scribal error here in the last paragraph is the inclusion of the word "sixth" in the list at the end of the second to last sentence.  Also note that Levi had no symbol for *.  He just wrote, for example, "the product 275539".

The smallest of these sums is 56 thousand and 700; the next is 60 thousand 750; the next is 63 thousand;  the next is 66 thousand and 150;  the next is 70 thousand and 875;  and the next is 75 thousand and 600; and the next is 77 thousand 750.  Now the excess of 77 thousand and 750 over the one before it, is 6 thousand and 150, and this is the excess of the fourth over the seventh.

> The number 6,150 should be 2,150.  This is a scribal error which appears correctly later.

Its excess over 70 thousand and 875, is 6 thousand 875, and this is the excess of the second over the seventh. Its excess over 66 thousand and 150, is 11 thousand and 600, and this is the excess of the first over the seventh. Its excess over 63 thousand is 14 thousand 750, and this is the excess of the sixth over the seventh. Its excess over 60 thousand 750, is 17 thousand 750, and this is the excess of the third over the seventh.

> This is another error which propagates. The number 17,750 should be 17,000, assuming 77,750 is correct.

Its excess over 56 thousand and 700, is 21 thousand and 50, and this is the excess of the fifth over the seventh. Accordingly, the excess of the fifth is the largest. The sum of all the numbers except the fifth, equals 56 thousand and 700. The excesses of the first, second, third, fourth and sixth over five times the seventh, equal 53 thousand and 125.

> The number 53,125 should be 52,375, even assuming that 77,750 is correct. This error propagated from the 17,750 error.

We subtract this from 56 thousand and 700, and what remains is 3 thousand and 575. We divide what remains by six, which is the number of numbers that make up the sum of 56 thousand and 700, to get 595 whole and 5 sixths of one. This is the seventh number. We add the excess of the sixth over the seventh, which is 14 thousand 750, to the seventh, and we get the sixth. This is 15 thousand 345 and 5 sixths of one. Accordingly, the fifth is 21 thousand 645 and 5 sixths; the fourth is two thousand 745 and 5 sixths; the third is 18 thousand 345 whole and 5 sixths of one; the second is 7 thousand 470 and 5 sixths of one; and the first is 12 thousand 195 and 5 sixths of one.

> The third value is an error. The number 18,345 should be 17,595, even assuming 77,750 to be correct. This error propagated from the 17,750 error.

You can check this if you wish.

> If you do check it, you will find that it is not correct. The original problem posed has no positive solutions! Hence even if all errors are corrected, there is no good reconstruction of this problem. Therefore, it is the only error in the whole problem section, that is almost surely Levi's. It may have been one of the main reasons why the problem was omitted from the second edition.

If the problem was about taking multiples of the rest such that the results were all one and the same; for example, we want to find four numbers such that the first with A multiples of the rest, equals the second with B multiples of the rest, equals the third with C multiples of the rest, and equals the fourth with D multiples of the rest.

> By "multiple", Levi means a number greater than one.

Then take the numbers that precede A, B, C and D, and call these E, G, H and I. That is, E is the number preceding A, G precedes B, H precedes C, and I precedes D. Let the product GHI be the sum of all the numbers except the first, the product EHI be the sum of all the numbers except the second, the product EGI be the sum of all the numbers except the third, and the product EGH be the sum of all the numbers except the fourth. Once you know all this, extract all the numbers in the previous fashion, each according to its proper place. You will find that these numbers are the numbers that were requested.

The proof is, that when you add the first to A multiples of the rest, the result equals the first plus as many of the product GHI as there are ones in A. This equals the first plus the rest plus as many of the product GHI as there are ones in A minus one, and this equals the product EGHI. Hence, the result equals the four numbers plus the product EGHI. Similarly, when you add the second to B multiples of the rest, the result is equal to the four numbers plus the product EGHI; and when you add the third to C multiples of the rest, the result is equal to the sum of all the numbers plus the product EGHI; and when you add the fourth to D multiples of the rest, the result is equal to the sum of all the numbers plus the product EGHI. This is what we wanted to explain.

Finally we claim that if the excess of the two numbers in our example, over two times the smallest, is greater than or equal to the sum of the three, then the problem is a fraud.

> Remember that EHI is the sum of all four numbers except the second, so that EHI is “the sum of the three” in the last paragraph, that is first+third+fourth.

For example, let the fourth number be the smallest, let K be the product EHI, let L be the excess of the first and the third numbers over twice the fourth, and assume that L is greater than or equal to K. We claim that the problem is a fraud, and that you can not find numbers that meet the required conditions.

> This “example” is the actually the beginning of a general proof. See the commentary at the end of the proof for a modern presentation.

The proof that you can not, is that if you could then let the numbers be M, N, O and P. Now divide the sum of M, O and P, by the number of ones in the product EHI, and let the parts found in this fashion be Q. There are as many Q’s in the sum of M, O and P, as there are ones in the product EHI. When we add B multiples of M+O+P, there are as many… as there are ones in the number G, because the number G is one less than B.

> This last sentence is missing some words, and in context of the rest of the proof, probably means to say that G(M+O+P) = B(M+O+P)-(M+O+P), because G = B-1.

But there are as many Q’s in M+O+P as there are ones in the product EGHI. Since M plus A multiples of the rest, equals N plus B multiples of the rest; and M plus A multiples of the rest, equals the number M+N+O+P plus as many N+O+P’s as there are ones in E; hence, there are as many Q’s in E times N+O+P, as there are ones in the product EGHI.

Thus there are as many Q’s in the product NOP, as there are ones in the product GHI.

There is a scribal error in the first line of the last paragraph, where "the product NOP" should be the number N+O+P.

Similarly, there are as many Q's in the number M+N+P, as there are ones in EGI, and there are as many Q's in the number M+N+O as there are ones in EGH. Thus there are as many Q's in the number M+O+P, as there are ones in K. It is clear by the previous explanation, that the number of Q's in the excess of M+O over two times P, is the same as the number of ones in L. But L is greater than or equal to K. Hence, the number M+O is bigger than the number M+O+P, but that is false. Hence, it is impossible to find numbers that meet the required conditions.

The proof in the last few paragraphs says that if A = E+1, B = G+1, C = H+1, D = I+1, and First+Third+Fourth = K = EHI ≤ L = First+Third-2(Fourth), then there is no solution. The proof is: Let Q = (M+O+P)/EHI, then (M+O+P)/Q = EHI and G(M+O+P)/Q = EGHI. Also, G(M+O+P) = B(M+O+P)-(M+O+P), because G = B-1. Since M + A (N+O+P) = N + B(M+O+P) and M+A(N+O+P) = M+N+O+P+E(N+O+P), then E(N+O+P) = EGHI(Q), and N+O+P = GHI(Q). Similarly, M+N+P = EGI(Q), M+N+O = EGH(Q), and M+O+P = KQ. Now LQ = M+O-2P and LQ ≥ KQ, so M+O-2P ≥ M+O+P and M+O > M+O+P, which is impossible.

Now I will give you an example. We want to find five numbers, such that the first plus three times the rest, equals the second plus three times the rest plus half, equals the third plus three times the rest plus a third of them, equals the fourth plus three times the rest plus 2 thirds of them, and equals the fifth plus four times the rest. By the preceding method, the sum of the rest without the first is 46 and 2 thirds; the sum of the rest without the second is 37 and a third; the sum of the rest without the third is 40; the sum of the rest without the fourth is 35; and the sum of the rest without the fifth is 31 and a ninth. The excess of the third over the first is 6 and 2 thirds; the excess of the second over the first is 9 and a third; the excess of the fourth over the first is 11 and 2 thirds; and the excess of the fifth over the first is 15 and 5 ninths. Hence, the excess of the second, third and fourth over 3 times the first, is 27 and 2 thirds. We subtract off the sum of all four, which is 31 and a ninth, and what remains is 3 and 4 ninths. We divide by the number of these numbers, and the first number is 31 of 36 parts of one; the second is 10 and 7 parts of 36; the third is 7 and 19 parts of 36; the fourth is 12 and 19 parts of 36; and the fifth is 16 and 15 parts of 36. You can check this if you wish.

The last example is completely correct. The rest of the problems contain no errors, and read relatively easily without extensive commentary.

**18. Problem:** We add one number to a second number; and the ratio of the result to a third number, is given. When we add the first number to the third number, the ratio of the result to the second number, is a second given number. One of the three numbers is known. What is each of the remaining numbers?

Given A and B, and one of X, Y or Z; find X, Y and Z such that (X+Y)/Z = A and (X+Z)/Y = B.

You already know how to find three numbers, that correctly meet these conditions, so extract them. Since you know one of the corresponding numbers to one of the three, you can extract the other corresponding numbers, and that is what was requested.

> The method is described in part one of the book. Namely, given (X+Y)/Z = A and (X+Z)/Y = B, then (AB-1)/X = (A+1)/Y = (B+1)/Z.

For example, when you add the first number to the second number, its ratio to the third equals 3 whole and 2 fifths and a seventh. When the first is added to the third, its ratio to the second equals 7 whole and 2 thirds and a fourth. The second number is thirty. We want to know: what is the value of each remaining number? First of all, extract three numbers, using the procedure described in part one of this book. Accordingly, subtract one, from the product of 3 whole and 2 fifths and a seventh, with 7 whole and 2 thirds and a fourth. This leaves 27 whole and a third of a seventh, which is the first number. Add one to 3 whole and 2 fifths and a seventh, to get 4 whole and 2 fifths and a seventh, which is the second number. Also add one to 7 whole and 2 thirds and a fourth, and the result you get is the third number, which is 8 whole and 2 thirds and a fourth. You already know that the number corresponding to the second number is thirty.

| First | Second | Third |
|---|---|---|
| 27 whole and a third of a seventh | 4 whole and 2 fifths and a seventh | 8 whole and 2 thirds and a fourth |
| 178 whole and 98 of 159 parts of one | Thirty | 58 whole and 281 of 318 parts of one |

Thus the number corresponding to the first is 178 whole and 98 of 159 parts of one; and the number corresponding to the third is 58 whole and 281 of 318 parts of one. These three numbers are what were requested, so investigate and find them.

It has already been explained how if you knew the sum of two of the latter numbers, you could find all the corresponding numbers, each in its proper place with respect to the known numbers. You could also do this if you knew the excess of one of the latter corresponding numbers over another one, or over the sum of two latter corresponding numbers; or if you knew the excess of the sum of two latter corresponding numbers over another one, or over the sum of another two latter corresponding numbers; or if you had any other information that provides enough knowledge to find the corresponding numbers, each matched to its proper place. Once you know these corresponding numbers, regardless of the source of the knowledge, we will explain why these corresponding numbers are the desired ones.

This is because the ratio of the first of the former numbers to the second of them, equals the ratio of the first of the latter numbers to the second of them; and the ratio of the second of the former numbers to the third of them, equals the ratio of the second of the latter numbers to the third of them. The value of this ratio equals the ratio of the first of the former numbers to the third, which equals the ratio of the first of the latter numbers to the third.  By adding these together, the ratio of the sum of the first and second former numbers, to the third, equals the ratio of the sum of the first and second latter numbers to the third.  But in our example, the ratio of the sum of the first and second former numbers, to the third, is 3 whole and 2 fifths and a seventh.  Hence, the ratio of the sum of the first and second latter numbers, to the third, is 3 whole and 2 fifths and seventh. Similarly, the ratio of the sum of the first and third former numbers, to the second, is 7 whole and 2 thirds and a fourth.  And use this as a model.

Another example:  the ratio of the sum of the first and second numbers, to the third, equals 3 fifths and a sixth;  the ratio of the sum of the first and third numbers, to the second, equals 2 whole and a third;  and the first number is 20.  We want to know:  what are the rest of the numbers?

Accordingly, extract the numbers that meet these conditions.  The first number according to what we explained is 2 fifths, and a third, and a third of a sixth.  The second number is one whole and 3 fifths and a sixth.  The third number is 3 whole and a third. Since the number corresponding to the first number is 20, then the number corresponding to the second number is 44 whole and 56 of 71 parts of one whole, and the number corresponding to the third number is 84 whole and 36 of 71 parts of one whole.  These are the requested numbers, and you can check this if you wish.

| First | Second | Third |
|---|---|---|
| 2 fifths and a third and a third of a sixth | One whole and 3 fifths and a sixth | 3 whole and a third |
| Twenty | 44 whole and 56 parts of 71 | 84 whole and 36 parts of 71 |

And it is worth pointing out that if the product that results from the ratio of the sum of the first and second to the third, with the ratio of the sum of the first and third to the second, is not greater than one whole;  then the poser of the problem erred.

We now suggest an explanation for this.  First we note, that given any two numbers, the product of the ratio of the first to the second, with the ratio of the second to the first, equals one whole.  This statement is true whether one of the given numbers is one part of the other, or is parts of the other.  First consider the case of it being one part, and the truth of our claim will soon be apparent.

The first case, of  "being one part" is when one number divides the other evenly.

For example, the given numbers are A and B, and B counts A as many times as there are ones in C.  Hence, the ratio of B to A is C, and the ratio of A to B is one of C parts of one.  When we multiply C whole with one of C parts of one, we get one as was explained previously.

Now consider the case where the smaller number is parts of the larger. We claim that the product of the ratio of one number to the other, with the ratio of the other number to the first, is one. In our example, let the smaller number be DE, and the larger number be B. Let DE equal G of C parts of B. Let EH equal one of C parts of B, so EH times G equals ED. Let the product of B with G equal IK. Then the ratio of HE to B equals the ratio of DE to IK. By exchanging them and swapping them around: the ratio of DE to HE, equals the ratio of IK to B. Also, the ratio of DE to B equals G times the ratio of HE to B; and the ratio of B to DE equals one of G parts of the ratio, IK to DE. But the product of the ratio of IK to DE, which is C, with the ratio of HE to B, equals the ratio of DE to IK, which equals one. Hence, the product of the ratio of DE to B, with the ratio of B to DE, equals one; because the factors suffice.

> In the proof in the last paragraph, Levi uses DE, HE and IK to mean single numbers in the geometric style of a line segment, as in Euclid. In Hebrew, he denotes this usage with the phrase "the number DE" in contrast to "the numbers DE", by which he means D+E. Although this is the only occurrence of this usage in the problem section, it appears very often in the proofs in the first volume of the book.
>
> The proof builds on the previous simpler case. Given B/DE, let DE = (G/C)B, EH = (1/C)/B, and B(G) = IK. Levi proves that DE/B = G(EH/B) and B/DE = (1/G)(IK/DE). Then (DE/B)*(B/DE) = G(EH/B)*(1/G)(IK/DE) = G(1/G)(EH/B)(IK/DE). Finally, since EH/B = 1/C and IK/DE = C, G(1/G)(EH/B)(IK/DE) = 1, by a reduction to the previous simpler case.

When this is understood, it is clear that the product resulting from the ratio of the sum of the first and second numbers to the third, with the ratio of the sum of the first and third numbers to the second, is greater than one. This is because the ratio of the sum of the first and third numbers to the second, is much greater than the ratio of the third to the sum of the first and second. However, the product resulting from the ratio of the sum of the first and second to the third, with the ratio of the third to the sum of the first and second, is one whole. Hence, the product resulting from the ratio of the sum of the first and second to the third, with the ratio of the sum of the first and third to the second, is greater than one whole.

**19. Problem:** The ratio of the second, to what remains from the third after subtracting off the first, is given number. The ratio of the third number to what remains from the second after subtracting off the first, is another given number. What is each one of the numbers that meet these conditions?

> Given A and B, find all X, Y and Z, such that Y/(Z-X) = A and Z/(Y-X) = B. Levi reduces this to Problem 18 by making an appropriate substitution.

It is appropriate to extract three numbers, according to the previous procedure, using the ratios given there. The first number will be the first number here. The sum of the first and the second numbers, will be the second number here. The sum of the first and the third numbers, will be the third number here.

For example, the ratio of the second to what remains from the third after subtracting off the first, is 3 whole and a third. The ratio of the third to what remains from the second after subtracting off the first, is the number 6. Now extract three numbers in the same fashion as before, using the given ratios. That is, the ratio of the sum of the first and second to the third, is 3 whole and a third; and the ratio of the sum of the first and third to the second, is 6 whole. Therefore, the first is 19, the second is 4 and a third, and the third is 7. Accordingly, the first number here is 19. Add 19 to the second, to get 27 whole and a third, and this is the second number here. Add 19 to the third, to get 26, and this is the third number here. And use this as a model. The reason for this is clear from the previous discussion.

And you can do here what you were able to do in the previous problem. That is, if one of the numbers is known to you, then once you have extracted the numbers corresponding to the unknown numbers, that meet these conditions, then you can figure out the other unknown numbers, and you can also take advantage of all the previous possibilities that provide enough knowledge to find the unknown numbers.

**20. Problem:** We add the first to the second, and the ratio of their sum to what remains from the third after subtracting off the first, is a given number. When we add the first to the third, the ratio of their sum to what remains from the second after subtracting off the first, is another given number. What is each of the numbers that meet these conditions?

> Given (X+Y)/(Z-X) = A and (X+Z)/(Y-X) = B, find all solutions X, Y and Z. Levi reduces this problem to Problem 18 and solves it with an appropriate substitution, as he did in Problem 19.

It is appropriate to extract three numbers in the previous fashion, using the given ratios. Half of the first number is the first number here. The sum of the second and this first, that is, half the previous first, is the second number here. The sum of the third and this first, is the third number here.

For example, the ratio of the sum of the first and second, to what remains from the third after subtracting off the first, is 4 whole and a half. The ratio of the sum of the first and third, to what remains from the second after subtracting off the first, is 5 whole. We want to know: what is each of the numbers that meet these conditions?

We extract these three numbers in the previous fashion using the given ratios. That is, the ratio of the sum of the first and second to the third, is 4 and a half; and the ratio of the sum of the first and the third to the second, is the number 5. Thus the first is 21 and a half, the second is 5 and a half, and the third is 6. Half of the first is 10 and 3 fourths, which is the first here. Add 10 and 3 fourths to the second, to get 36 and a quarter, which is the second here. Add 10 and 3 fourths to the third, to get 16 and 3 fourths, which is the third here. And use this as a model. The reason for this is clear from the previous discussion.

After you know the numbers that meet these conditions, you can use them to find the corresponding unknown numbers as we did before. That is, with information about one of the corresponding numbers, the sum of two such numbers, or all other variations of information mentioned earlier, we can derive the corresponding unknown numbers to the three given numbers.

**21. Problem:** The first number plus a given part of the sum of the second and third numbers, equals the second number plus some other given part of the sum of the first and third numbers. It also equals the third number plus another given part of the sum of the first and the second numbers. One of the numbers is given. What are the rest of the numbers?

> Given A, B and C, and one of X, Y or Z, find X, Y and Z, such that
> $X+(Y+Z)/A = Y + (X+Z)/B = Z + (X+Y)/C$.
>
> This is a special case of the main result in Problem 17. It is comparatively concise and easy to read. The only thing here not appearing in Problem 17, is a brief comment at the end, similar to the comments in Problems 18-20, on how to use the general solution to find particular solutions, when given a variety of information about the unknown numbers.
>
> The general solution is: $X = C+(B-A)+(A-2)BC$, $Y = X + 2(B-A)(C-1)$ and $Z = Y + 2(A-1)(C-B)$. The particular solutions are computed by setting up the appropriate proportions with the general solution.

It is appropriate to first extract the numbers that meet these conditions, according to the method discussed in part one of this book. From your knowledge of one of the corresponding unknown numbers, you can then learn all the corresponding unknown numbers, each with respect to its corresponding known number. These numbers are what were requested.

For example, the first with a fourth of the rest, equals the second with a sixth of the rest. This also equals the third with a ninth of the rest. As was already explained in part one of this book, the order of these three numbers is according to the order given here. That is, the first is the one added to the largest fraction of the rest, and the third is the one added to the smallest fraction of the rest.

> That is, he assumes that $1/A > 1/B > 1/C$.

I am reminding you about this, so that you do not get confused about their order. The second number is 20. We want to know: what are the rest of the numbers? We already can derive that of the corresponding numbers meeting these conditions: the first is 119, the second is 151, and the third is 169. The number corresponding to the second is 20. We extract the other corresponding numbers based on this ratio. Accordingly, the number corresponding to the first is 15 whole and 115 of 151 parts of one whole, and that is the first number here. The number corresponding to the third is 22 whole and 58 of 151 parts of one, and that is the third number here. These are the requested numbers, and you can check this if you wish.

This is right because the latter numbers are in proportion with the former numbers. Therefore, the ratio of the first of the former numbers to the second of them, equals the ratio of the first of the latter numbers to the second of them. Accordingly, the ratio of the first of the former numbers to the third, equals the ratio of the first of the latter numbers to the third. By adding these together, we get that the ratio of the first of the former numbers to the sum of the second and third of them, equals the ratio of the first of the latter numbers to the sum of the second and third of them. Accordingly, the ratio of the first of the former numbers, to a quarter of the sum of the second and third of them, equals the ratio of the first of the latter numbers, to a quarter of the sum of the second and third of them. By adding these together, we get that the ratio of the first of the former numbers plus a quarter of the second and third of them, to the first of the former numbers, equals the ratio of the first of the latter numbers plus a quarter of the second and third of them, to the first of the latter numbers.

Similarly, the ratio of the second of the former numbers plus a sixth of the third and first of them, to the second of the former numbers, equals the ratio of the second of the latter numbers plus a sixth of the third and first of them, to the second of the latter numbers. Similarly, the ratio of the third of the former numbers plus a ninth of the rest of them, to the third of the former numbers, equals the ratio of the third of the latter numbers plus a ninth of the rest of them, to the third of the latter numbers.

By exchanging them, the ratio of the first of the former numbers plus a fourth of the rest of them, to the first of the latter numbers plus a fourth of the rest of them, equals the ratio of the first of the former numbers to the first of the latter numbers; the ratio of the second of the former numbers plus a sixth of the rest of them, to the second of the latter numbers plus a sixth of the rest of them, equals the ratio of the second of the former numbers, to the second of the latter numbers; and the ratio of the third of the former numbers plus a ninth of the rest of them, to the third of the latter numbers plus a ninth of the rest of them, equals the ratio of the third of the former numbers, to the third of the latter numbers.

However, the ratio of the first of the former numbers to the first of the latter numbers, equals the ratio of the second to the second, and equals the ratio of the third to the third. Hence, the ratio of the first of the former numbers plus a fourth of the rest of them, to the first of the latter numbers plus a fourth of the rest of them, equals the ratio of the second of the former numbers plus a sixth of the rest of them, to the second of the latter numbers plus a sixth of the rest of them, which equals the ratio of the third of the former numbers plus a ninth of the rest of them, to the third of the latter numbers plus a ninth of the rest of them. By exchanging, they will all be related. But the former are all equal so the latter are all equal. And use this as a model.

Thus, with any knowledge of the unknown numbers, you can extract all the unknown numbers corresponding to the three known numbers. That is, if you knew the excess of one unknown over another, or the sum of two unknowns, or any similar kind of knowledge, you could derive all the missing numbers, as we explained previously. Comprehend and solve.

**(Ed. 1)** The author writes: the sixth section of this volume is complete, and with its completion, the book is complete. The praise goes exclusively to God. Its completion was at the start of Nissan of the 81st year of the 6th millenium, when I reached the 33rd of my years. Bless the Helper.

**(Ed. 2)** The sixth section of this volume is complete, and with its completion, the book is complete. The praise goes exclusively to God. Its completion was in the month of Elul of the 82nd year of the 6th millenium. Bless the Helper.

## III. A Critical Edition of Problems 16-21

We present a critical edition in Hebrew of Problems 16-21 using all twelve extant mss. For further discussion, please see part one of this critical edition, in the previous issue of this journal.

## Appendix: List of the Theorems from Part One of *Maaseh Hoshev* in the Two Editions, with Brief Notes.

The regular numbering is from the first edition, and the numbering in parentheses is from the second.

--- 1. The product of 2 numbers a and b, is a added to itself b times.
(No proof).

(1) 2. The product of a number a and another $b = b_1 + b_2 + ... + b_n$ is $ab_1 + ab_2 + ... + ab_n$.
(Proof just unravels the theorem using 1).

(2) 3. The product of 2 numbers $a = a_1 + a_2 + ... + a_m$ and $b = b_1 + b_2 + ... + b_n$ is
$a_1b_1 + a_1b_2 + ... + a_1b_n + a_2b_1 + a_2b_2 + ... + a_2b_n + ... + a_mb_1 + a_mb_2 + ... + a_mb_n$.
(Proof just unravels the theorem using 2).

(3) 4. The product of a number $a = b + c$ with b is equal to $b^2 + bc$.
(Corollary of 2, proof is immediate).

(4) 5. $(a/2 + b)$ squared is equal to $(a+b)b + (a/2)^2$
(Proof uses 3).

(5) 6. $(a+b)^2 = a^2 + b^2 + 2ab$.
(Proof uses 3).

(6) 7. $(a+b)^2 = a(a+b) + ab + b^2$
(Proof uses 3).

--- 8. If $a = b+c$, then either $(a/2)^2 = bc + (b – a/2)^2$ or $(a/2)^2 = bc + (c – a/2)^2$.
(Proof uses 3. Does not consider the negative case.)

(7) 9. $a(bc) = b(ac) = c(ab)$.
(Proof uses 1).

(8) 10. $a(bcd) = b(acd) = c(abd) = d(abc)$.
(Proof uses induction and 9, and explicitly implies a more general theorem, that you can take any n numbers, and their product will be the same as the product of any n-1 terms times the remaining term).

(9) 11. $a(bcd) = (ac)(bd)$.
(Proof uses 10, and explicitly implies a more general theorem, that you can take any n numbers, and their product will be the same as the product of any n-2 terms times the product of the remaining two terms).

(10) 12. $a(b_1)(b_2)...(b_n) = b_1(a)(b_2)(b_3) ... (b_n) = b_2(a)(b_1)(b_3) ... (b_n) = ...= b_n(a)(b_1)(b_2)...(b_{n-1})$.
(Completely generalizes 10 and 11).

(11) 13. The ratio $((a_1)(a_2)...(a_n))/((b_1)(b_2)...(b_n)) = (a_1/b_1) (a_2/b_2) ... (a_n/b_n)$.

(12) 14. The ratio $((a_1)(a_2)...(a_n))/((b_1)(b_2)...(b_n))$ equals the product of $a_i/b_j$, where each i and j between 1 and n appears exactly once.

--- 15. If a is relatively prime to $b = (b_1)(b_2)...(b_n)$, then a is relatively prime to $b_i$, for all i from 1 to n.

--- 16. A number a that is relatively prime to all the integers less than $\lceil \sqrt{a} \rceil$, is prime.
($\lceil \sqrt{a} \rceil$, is literally: $\sqrt{x}$ where x is the first square larger than a.)

--- 17. If one takes a fraction of a given number, and then a fraction of the remainder and continues arbitrarily, then the final remainder will be the same no matter what order the fractions were taken. Also, the sum of all the pieces taken will be the same.

--- 18. If one takes a fraction of a given product, and then a fraction of the remainder and continues arbitrarily, then the final remainder will be the same no matter what order the fractions were taken.

(13) 19. The number of terms in the sum 1+ 2+ ... +n, is equal to the number of 1's in n.

(14) 20. The number of odd terms in the sum 1+2+...+2n is equal to the number of even terms.

(15) 21. In the sum, n + (n+1) + (n+2) + ... + (n+m), the last term is m greater than the first.

(16) 22. In the sum (n-m) + (n-m+1) + ... + n + (n+1) +...+(n+m), the last term exceeds the middle term by the amount that the middle term exceeds the first term.

(17) 23. In the sum (n-m) + (n-m+1) + ... + n + (n+1) +...+(n+m), the first term is odd iff the last term is odd.

(18) 24. If a-1 = c-b, then a+b = c+1.

(19) 25. If a-c = c-b then a+b = 2c.

(20) 26. 1+2+...+n, where n is even, is equal to (n/2)(n+1).
(Literally, half the number of terms times the number of terms plus 1. Proof works from outside in, in pairs showing that each pair sums to n+1,and there are n/2 pairs).

(21) 27. 1+2+...+n, where n is odd, is equal to ((n+1)/2)n.
(Literally the middle term times the number of terms. Proof works from inside out, in pairs showing that each pair sums to twice the middle term.)

(22) 28. 1+2+...+n, where n is odd, is equal to (n/2)(n+1).
(Literally, half the last term times the number after the last term.
Proof uses proportions, algebra like idea and 21).

(23) 29. $1+3+5+...+(2n-1) = n^2$.
(Literally, the square of the middle term. Proves it first for an odd number of terms, then an even number).

(24) 30. $(1+2+...+n)+(1+2+...+n+(n+1)) = (n+1)^2$.

(25) 31. $2(1+2+...+n) = n^2+n$.
(Proof uses 30).
(Corollary: The sum $1+2+ .. + n = n^2/2 + n/2$.)

(26) 32. $1 + (1+2) +(1+2+3) + ... + (1+2+...+n) = 2^2+4^2+6^2+...+n^2$, n even; and $1^2+3^2+5^2+...+n^2$, n odd.
(Proof uses 30).

(27) 33. $(1+2+3+...+n)+(2+3+4+...+n)+ ... + n = 1^2+2^2+3^2+...+n^2$
(Proof uses a counting argument).

(28) 34. $(1+2+3+...+n)+(2+3+4+...+n)+ ... + n+1+(1+2)+(1+2+3) + ... + (1+2+...+(n-1)) = n(1+2+3+...+n)$
(Proof uses a counting argument).

(29) 35. $(n+1)^2+n^2 - (n+1+n) = 2n^2$

(30) 36. $(1+2+3+...+n)+(2+3+4+...+n)+ ... + n - (1+2+3+...+n) =$
$2(2^2+4^2+6^2+...+(n-1)^2)$, n-1 even; and $2(1^2+3^2+5^2+...+(n-1)^2)$, n-1 odd.
(Proof uses 33 and 35).

(31) 37. $n(1+2+3+...+(n+1)) = 3(1^2+3^2+5^2+...+n^2)$, n odd; and $3 (2^2+4^2+6^2+...+n^2)$, n even.
(Proof uses 32, 34 and 36).

(32) 38. $(n-(1/3)(n-1)) (1+2+3+...+n) = 1^2+2^2+3^2+...+n^2$
(Proof uses 32, 33, 34 and 37).

(33) 39. $(n^2-n)/2 = (1+2+...+(n-1))$
(Proof uses 30).

(34) 40. $(n^2-n)/2 + n = (1+2+...+n)$
(Proof uses 30).

(35) 41. $(1+2+3+...+n)^2 = n^3 + (1+2+3+...+(n-1))^2$
(Proof uses 30 and 6).

(36) 42. $(1+2+3+...+n)^2 = 1^3+2^3+3^3+...+n^3$
(Proof by induction using 41).

(37) 43. Let $m = 1+2+3+...+n$, then $1^3+2^3+3^3+...+n^3 = 1+3+5+...+(2m-1)$.

(38) 44. $ab+a = (b+1)a$, and $ab+b = (a+1)b$.

(39) 45. Given $a<b<c$, then $c(b-a)+a(c-b) = b(c-a)$.

(40) 46. Given $2<a<b$, then $2(a-2)(b-1)+b+(a-2)+(b-a) = 2(a-1)(b-1)$.
(Minor differences in the two editions).

(41) 47. Given $a<b$, then $ba+(b-a) = (a-1)(b-1)+b+(b-1)$.

(42) 48. Given $2<a<b$, then $2(b-1)(a-2)+b+(a-2)+(b-a) = (a-1)b+(a-2)(b-1)+(b-a)$.
(Minor differences in the two editions).

(43) 49. Given $a<b<c$, and $d=a-2$, then $2(c-1)(b-a)+cd+(c-1)d+c+(b-a)+(c-b) = 2(b-1)(c-1)$.
(Minor differences in the two editions).
(Generalizes 46).

(44) 50. Given $a<b<c$, and $d=a-2$, then $2(c-1)(b-a)+(cb)d+(c-b)(a-1)+c+(b-a) = (b-1)(c-1)a$.
(Major differences in the two editions).

(45) 51. Given $a<b<c$, then $(c-1)(b-a)+c+(b-a) = c(b-a+1)$.

(46) 52. Given $a<b<c$, then $(c-1)(b-a)+(a-1)(c-b)+c+(b-a) = b(c-a+1)$.

(47-8) 53. Find x,y,z such that $x+(y+z)/a = y+(x+z)/b = z+(x+y)/c$, where $a<b<c$...

(49) 54. Find x, such that fractions of x minus smaller fractions of x, equals a.

(50) 55. Given that x + big fractions of x is smaller than y + small fractions of y, find z, such that x + big fractions of z+y, equals y + small fractions of z+x.

(51) 56. Find x, such that given fractions of x, equal other given fractions of a.

(52) 57. Find x,y, such that x + fractions of y, equals y + other fractions of x.

(53) 58. Find x,y,z, such that x+z = ay and y+z = bx.

(54) 59. $(a_1)^2(a_2)^2(a_3)^2...(a_n)^2 = ((a_1)(a_2)(a_3)...(a_n))^2$

(55) $(ab^2)(ac^2) = (abc)^2$

(56a) 60. $(a_1)^3(a_2)^3(a_3)^3...(a_n)^3 = ((a_1)(a_2)(a_3)...(a_n))^3$

(56b) $(ab^3)(a^2c^3) = (abc)^3$

(57) 61. $ab^2+ba^2 = ab(a+b)$

(58) 62. $(a+b)^3 = a^3+3ab(a+b)+b^3$

(59) 63. $P_{n+1} = (n+1)P_n$, where $P_n$ is the number of different ways to order n elements.
(Corollary: $P_n = n!$).

(60) 64. $P_{n,2} = n(n-1)$, where $P_{n,m}$ is the number of ways to order m elements out of n.

(61) 65. $P_{n,m+1} = P_{n,m}$ (n-m).
(Corollary: $P_{n,m} = n!/(n-m)! = n(n-1)(n-2)...(n-(m+1)))$.

(62) 66. $P_{n,m} = C_{n,m}P_m$, where $C_{n,m}$ is the number of ways to choose m elements out of n without regard to order.

(63) 67. $C_{n,m}= P_{n,m} /P_m$

(64) 68. $C_{n,n-m} = C_{n,m}$.

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