

# The Missing Problems of Gersonides – A Critical Edition – Part I

By

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## Abstracts

Gersonides' *Maaseh Hoshev*, (*The Art of Calculation*), is a major work known for its early use of rigorous combinatorial proofs and mathematical induction. There is a large section of problems at the end of the book, with the theme of proportions, which until now remained unpublished. I present a critical edition of this material. I also uncover a previously unknown second edition of *Maaseh Hoshev*. The material is appropriate for creative pedagogy and provides economic details of the author's culture and environs. This article presents the first fifteen problems and a subsequent article presents the rest.

מעשה חושב לרלב"ג הוא חיבור חשוב הידוע הודות לשימושו המוקדם בהוכחות קומבנטוריות מדויקות ובאינדוקציה מתימטית. בסוף הספר מופיע פרק גדול של שאלות שנושאן יחסים, שטרם פורסם. אני מציג הוצאה ביקורתית של פרק זה. החומר חושף מהדורה שנייה לא ידועה של מעשה חושב. החומר מתאים לפדגוגיה יצירתית ומספק פרטים כלכליים על אורח חייו וסביבותיו של המחבר. במאמר זה מובאות חמש עשרה הבעיות הראשונות והיתר יופיעו במאמר נוסף.

*Le Maase Hoshev (L'Art du Calcul)* de Gersonide est un ouvrage majeur connu pour son usage precoce de preuves combinatoires rigoureuses et de l'induction mathematique. La fin de l'ouvrage comporte une section importante consacree a une serie de trente problemes sur le theme des proportions qui etait restee inedite jusqu'a ce jour et don't je presente pour la premiere fois une edition critique. Je revele egalement l'existence d'une seconde edition de *Maase Hoshev*. Le materiau peut servir de base a des projets de pedagogie creative et offre des details sur la culture et l'environnement de l'auteur. Cet article presente les quinze premiers problemes de *Maase Hoshev*; un second article en presente les quinze autres.

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## I. Introduction

A critical edition with translation and commentary of a collection of heretofore unpublished problems, by the noted medieval scientist, philosopher and mathematician, Levi ben Gershon, a.k.a. Gersonides, is presented. The publication of these problems coincides with my discovery of a second edition of *Maaseh Hoshev*, the book which contains these problems.

The mathematical theme underlying all the problems is the use of proportions. The problems are often presented in a practical manner, and thereby reflect and confirm historical prices, measures, exchange rates for different coins, wages, and types of goods and services in 14<sup>th</sup> century Provence. The problems also provide useful material for creative pedagogy.

The critical edition is presented in two articles. This first article includes the first 15 problems. The second article, to appear in a subsequent issue of this journal, includes Problems 16-21. These later problems contain a number of difficult sections and errors.

## II. Levi ben Gershon, a.k.a. Gersonides

Levi ben Gershon (1288-1344), rabbi, philosopher, astronomer, scientist, biblical commentator and mathematician, was born in Provence (South France) and lived there all his life. Through his writings, he distinguished himself as one of the great medieval scientists and a major philosopher. He wrote more than a dozen books of commentary on the Old Testament, a major philosophical work called *MilHamot Adonai*, Wars of God, a book on logic, four treatises on mathematics, and a variety of other scientific and philosophical commentaries. *MilHamot Adonai* has a section on trigonometry and a long section on astronomy, including the invention of the Jacob's Staff, a device to measure angles between heavenly bodies used for centuries by European sailors for navigation, a discussion of the *camera obscura*, and original theories on the motion of the moon and planets. A complete bibliography on Levi and his work can be found in [18].

There is little information about Levi's personal life except that he came from a family of scholars and learned men. Although the details of these relationships are not firmly established, for example, one can find no less than three different claims (Levi ben Abraham ben Hayim, Gershon ben Shlomo, and NaHmonides) for the identity of his grandfather, it is well known that his home of Provence in southern France was a place that supported productive Jewish scholarship in the 14<sup>th</sup> century.

Life for the Jews in 14th century Provence was relatively good. The Pope, who was then at Avignon, protected them from oppression, which was not the case in the northern parts of France. There were 15,000 Jews among the two million general population of 14<sup>th</sup> century Provence. Jews worked as money lenders, physicians, craftsman and merchants.

Levi was highly regarded by the Christian community as a scientist and mathematician. He is referred to by them as Leo Hebraus, Leo de Balneolis or Maestro Leon.

His book on trigonometry was dedicated to Pope Clement VI and his book, *On Harmonic Numbers*, was commissioned by Phillip of Vitry, Bishop of Meaux. Both of these works were translated from Hebrew to Latin in Levi's lifetime. Levi wrote only in Hebrew, and read Hebrew translations of Latin and Arabic works, readily available to him.

Despite his originality and reputation, much of Levi's scientific and mathematical work did not heavily influence his successors. It is not clear to what extent he played a role in the transmission of Hellenistic mathematics from the Arab world to western Europe.

### **Levi's Mathematics**

Levi's mathematics comprises four major works.

A commentary on Euclid was completed in the early 1320's, with ready access to Hebrew translations. Included is an attempt to prove the fifth postulate. The content is discussed in [23].

*De Sinibus, Chordis et Arcubus*, a treatise on trigonometry in *MilHamot Adonai*, was completed in 1342 and dedicated to the Pope. There is a translation of this work into German [6] and an expository article on its content [10].

*De Numeris Harmonicis*, completed in 1343, was commissioned by Phillip de Vitry, Bishop of Meaux, and immediately translated from Hebrew into Latin. Philip was a musicologist interested in numbers of the form  $2^n 3^m$  called harmonic numbers. Levi proves that the only pairs of harmonic numbers that differ by one are (1, 2), (2, 3), (3, 4) and (8, 9). The book is relatively short and the original Hebrew is lost. The Latin version and commentary can be found in [5]. A summary of the contents can be found in [7].

*Maaseh Hoshev*, Levi's first and largest mathematics book dated 1321, contains the missing problems presented here. It is known for its early use of mathematical induction. Translations and work discussing its content include [5; 7; 16; 17; 20; 21; 26; 31; 32].

### **III. *Maaseh Hoshev*, The Art of Calculation**

*Maaseh Hoshev*, The Art of Calculation, dated 1321, is Levi's first book on mathematics and his second book overall. (Levi's first book is on logic and is dated 1319 [24]). The title comes from the biblical book of Exodus, where the phrase is used in a number of different places to describe the type of work necessary in constructing the tabernacle. Although it literally means "A Work of Calculation", its true meaning is more subtle. It is translated below as "cunning work".

#### **Exodus 29:2-3**

*And he made the ephod of gold, blue, purple, and scarlet, and fine twisted linen. And they did beat the gold into thin plates, and cut it into wires, to work it in the blue, and in the purple, and in the scarlet, and in the fine linen, with **cunning work**.*

This *cunning work* is in contrast with other types of work in the construction of the tabernacle, which are referred to by the skill required, such as *maaseh rokem*, the work of an embroiderer, or *maaseh oreg*, the work of a weaver, or *maaseh harash*, the work of an engraver, or *maaseh avot*, the work of a braider. The intention is that *maaseh hoshev* denotes a skill that requires more than just technical craftsmanship. It is the skill of an architect, requiring thought, cunning, planning and calculation.

The title is also a play on words for theory and practice, *Maaseh* corresponding to practice and *Hoshev* corresponding to theory. Levi writes in his introduction to the book:

*“It is only with great difficulty that one can master the art of calculation, without knowledge of the underlying theory. However, with the knowledge of the underlying theory, mastery is easy... and since this book deals with the practice and the theory, we call it Maaseh Hoshev”.*

*Maaseh Hoshev* is a major work in two parts. Part one is a collection of 68 theorems and proofs in Euclidean style about arithmetic, algebra and combinatorics. Part two contains algorithms for calculation and is subdivided into six sections:

- a. Addition and Subtraction
- b. Multiplication
- c. Sums
- d. Combinatorics
- e. Division, Square Roots, Cube Roots
- f. Ratios and Proportions.

A large collection of problems appears at the end of Section f in part two. Not including this collection of problems, parts one and two are about the same size, and the problems are about a third the size of each part. The text and the problem section of part two often refer back to the theorems in part one. Levi lists Euclid, books 7-9, as prerequisite reading.

### **Lange’s Critical Edition**

Lange wrote a critical edition of *Maaseh Hoshev* in 1909 with an introduction, translation and commentary in German [20]. It ignores the problem section completely and includes nothing about the second edition.

Lange had knowledge of only four mss. In his introduction, he mentions that his main source was the Vienna ms. because it is the most complete, and that he used the Munich 68 ms. to help distinguish between letters in the Vienna ms. that look similar. He adds that he never actually saw the Paris ms., but was able to learn some details about it through a friend. It should be noted that the Paris and Vienna mss. include all the missing problems, while Munich 36 has none, and Munich 68 contains just three. Also note that the Vienna and Munich 68 mss. represent the same family of mss.

## **Lange and the Missing Problems**

As far as the problem section is concerned, Lange describing the Vienna ms., writes:

*“Wir mußten sie einer späteren Veröffentlichung vorbehalten, da es aus gewissen Gründen nicht möglich war, sie jetzt mit zu veröffentlichen.”*

*We had to leave the problems for a future publication, because of specific reasons, it was not possible to publish them here.*

Lange never published a subsequent paper on the missing problems. I speculate that this was because:

1. He was unable to work his way through the maze of Problem 17 with only one source.
2. There are a number of errors in other problems, which are hard to resolve especially without being able to distinguish between the similar looking letters present in the Vienna ms. He was unable to distinguish these letters because the Munich ms. he used for this purpose, is missing most of the problem section.

## **Two Editions of *Maaseh Hoshev***

In the literature, *Maaseh Hoshev* is dated 1321, however, this information is incomplete. There are, in fact, two editions of *Maaseh Hoshev*. One is the basis of Lange's 1909 critical edition, and the other is dated one and a half years later. The editions can be distinguished by their colophons which date the edition either as 1321 or 1322. When the colophon is missing, the edition can be identified by other distinguishing features.

The major difference between the two editions is the omission in the second edition of two problems, one on the solution of certain quadratic equations, and the other, a very long problem on the solution of certain simultaneous linear equations. The theorems in part one related to these problems are also omitted. There are other minor additions, changes and omissions in both the theorems of part one and the algorithms and problems of part two. The theorems in part one of the second edition are renumbered due to these changes, and these new numbers are used in part two of the second edition for referencing purposes.

## **Why a Second Edition?**

Levi has no special preface, nor does he give any explanation for the second edition, but it is clear that the second edition is simply a reworking of the text for the ease of the reader. The differences in the second edition are primarily not in content but in organization and presentation. In the second edition, Problems 16 and 17 are omitted, and Problems 14 and 15 are completely rewritten. Problems 14 and 15 of the second edition present the same material as Problems 14 and 15 of the first edition but use techniques that are more easily generalized. Problem 16 deals with the solution of quadratic equations and is the only one of

all 21 problems that does not use proportions. The problem does not fit in naturally with the rest of the section. Problem 17 also differs from the rest of the problems. It is about ten times the length of any of the other problems, and includes very long tedious proofs with difficult error-prone notation rendering them all but impossible to read. What's more, Problem 21 reviews the main points contained in Problem 17 in a much more condensed and simple fashion. Finally, the only real error in the whole book, that is, one which is clearly not the result of a careless or ignorant scribe, and one which cannot be corrected or reconstructed, is contained in Problem 17.

A final and important difference between the two editions, is that the colophon of the first edition appears preceded by the words "The author writes:", and has a self reference to Levi's age at the date given; while the colophon of the second edition has neither. It is Levi's consistent style to start all his work with the words "Levi says", thereby identifying himself as the author. It is possible that Levi left out the personal references in the colophon of the second edition for no special reason except that it *was* a second edition. However, it is also possible that he wanted to give credit to students who may have helped in the editing process. Glasner's work [13] implying a school of Levi's students would support this possibility.

The two colophons with their translations are shown below.

<Insert Vienna colophon here>

*The author writes: the sixth section of this volume is complete, and with its completion, the book is complete. The praise goes exclusively to God. Its completion was at the start of Nissan of the 81<sup>st</sup> year of the 6<sup>th</sup> millenium, when I reached the 33<sup>rd</sup> of my years. Bless the Helper.*

**Figure 1. Colophon from Vienna Ms. of *Maaseh Hoshev*, representing the first edition.**

<Insert Moscow colophon here>

*The sixth section of this volume is complete, and with its completion, the book is complete. The praise goes exclusively to God. Its completion was in the month of Elul of the 82<sup>nd</sup> year of the 6<sup>th</sup> millenium. Bless the Helper.*

**Figure 2. Colophon from Moscow 1063 Ms. of *Maaseh Hoshev*, representing the second edition.**

## **Twelve Extant Manuscripts**

There are 12 extant mss. of *Maaseh Hoshev*, listed below according to their current locations. Nine represent the first edition and three represent the second edition. In the first edition there are 21 problems and in the second edition there are 19. The first 13 problems, numbered 1-13, and the last four, numbered 18-21, are the same in both editions. Problems 14-15 in the second edition are a new presentation of the material in problems 14-15 of the first edition. Problems 16-17 of the first edition do not appear in the second edition.

The information presented here on dates, scripts and completeness, is based on a combination of my own research, the catalogs at the Institute of Microfilmed Hebrew Manuscripts in the Jewish National University Library, and personal communication with Malachi Beit-Arie', Ludwig Jesselson Professor of Codicology and Paleography at the Hebrew University of Jerusalem.

I list the mss. below in order of completeness with respect to the problem section of *Maaseh Hoshev*. The abbreviation used for reference in the critical apparatus appears in parenthesis to the right of each location. For each ms., information is given about the date, margins, similarity to other mss., and content of missing or omitted material. In general, when I say "missing" I mean that the material was included by the scribe, but the pages were later lost, while "omitted" means that the material was not included by the scribe. Within edition one, I identify two families of mss. by their consistent agreement on dozens of phrases and minor errors. A stemma summarizing the notes below appears at the end. Note that the information for the stemma is extracted mainly from the problem section of each ms. Therefore the mss. in which much of the problems are missing or omitted cannot be reliably identified until the completion of a new complete critical edition for the whole book.

#### **Edition One: Nine Extant Mss.**

Part one: 68 Theorems and Proofs.  
Part two: Six Sections of Algorithms.  
Problems: 21 Problems.  
Colophon: Nissan 5081 (Spring 1321)  
The Author Writes...When I Reached My 33<sup>rd</sup> Year

**Vatican (Va)** Missing all of the Problem Section.  
Missing the end of part one, all of part two, the problem section, and the colophon. Italian script. Estimated date – late 14<sup>th</sup> century.

**Munich 36 (Mu36)** Missing all of the Problem Section.  
Omits the last third of part one, the last third of part two, all of the problem section and the colophon. This ms. is only nine pages of very small script, and bound with a larger book of mathematics including a Hebrew translation of Euclid. It seems to be someone's personal abridged version. Note that the most difficult and interesting parts of the book are missing, including proofs by induction, solutions of certain simultaneous linear equations, all the combinatorics, square and cube root extraction, and proportions. No margin notes. Sephardic script. Estimated date – 15<sup>th</sup> century.

**Munich 68 (Mu68)** Contains only Problems 18-20.  
Omits the last one and a half sections of part two and most of the problem section, then continues with the last few problems of the problem section, and concludes with the first edition colophon. The omissions here are less consistent thematically than in Munich 36, and seem more likely to correspond to a block of missing pages in the original copied source. The ms. is similar to Vienna, Pm2462 and London. No margin notes. Ashkenazic script. Dated 1552.

**New York (NY)** Contains Problems 1- middle of 17.

Missing the first 90% of part one, the end of the problem section, and the colophon. The last page (two sides) in the volume is misplaced and is really part of section five. A fragment of *Sefer Hamispar* by Abraham Ibn Ezra is bound in front. It was purchased by Ephraim Deinnard, a well known Hebrew manuscript dealer, in the early 1900's in Hebron. It is quite different from the other mss. and not faithful to the particular wording of the original, consistently omitting or replacing words and taking great liberty in paraphrasing. It also omits all figures not even leaving empty space. It is similar to Pm2271 and Pa in its errors. Very few margin notes. Sephardic script with Byzantine elements. Estimated date – late 15<sup>th</sup> / early 16<sup>th</sup> century.

**London (Lo)** Contains Problems 1- middle of 17  
(one more paragraph than NY).

Missing the very end of the problem section and the colophon. It is the most reliable of the London, Vienna, Pm2462, Mu68 family. There is a large number of margin comments and corrections, many by Mordecai Finzi. Sephardic script. Estimated date – late 14<sup>th</sup> / early 15<sup>th</sup> century.

**Parma 2462 (Pm2462)** Contains Problems 1- middle of 20.  
Missing the very end of the problem section and the colophon. Believed to have been produced in Italy in the early 1400's (watermark identical to a 1426 ms). Part of the London, Pm2462, Mu68, Vi family. It is likely that Parma 2462 was an earlier source because it has fewer errors and contains some phrases that are omitted in Vienna and London. No margin notes. Semi-Square Provençal script. Estimated date – late 14<sup>th</sup> / early 15<sup>th</sup> century.

**Vienna (Vi)** Contains all first edition problems.  
Complete first edition. Used as Lange's main source in the 1909 critical edition. It is the least reliable of the London, Vienna, Pm2462, Mu68 family. This ms. is not likely the source of the other mss. in the family, because it contains some hard to distinguish letters. If it was the source of the other mss. in the family, one would expect more errors in these letters in the other mss. than are actually found. No margin notes. Ashkenazic script. Dated 1462.

**Paris (Pa)** Contains all first edition problems.  
Complete first edition but the colophon is replaced with a standard short version with no dates or details. This ms. was brought to the library of the Oratory in 1620, and remained there until the Revolution, when it was moved to the Bibliotheque Nationale. All figures are omitted and no empty space is left. It shares features of both the NY, Pm2271 family and the London, Pm2462, Vienna, Mu68 group, but is generally much closer to the former. Very few margin notes. Sephardic script with Italian elements. Estimated date – 15<sup>th</sup> century.

**Parma 2271 (Pm2271)** Contains all first edition problems.  
Complete first edition. Believed to have been produced in Provence in the late 1300's making this the oldest extant ms. The innermost bifolium of the last quire is bound in backwards, resulting in the third and fourth to last pages being interchanged. This implies that the last eight sides of the ms. are numbered 34125678 instead of 12345678, where the eighth (and last) side is blank. Part of the Pa, NY family, but more reliable than either.



Figures are in the margins with a few omitted. No margin notes. Provençal script.  
Estimated date – late 14<sup>th</sup> century.

### **Edition Two: Three Extant Mss.**

Part one:	64 Theorems and Proofs. 2 Additions, 6 Omissions and Some Minor Variations
Part two:	Six Sections of Algorithms with Minor Additions and Omissions
Problems:	19 Problems. 2 Omissions, 2 Major Variations and Some Minor Variations
Colophon:	Elul 5082 (Fall 1322) No Reference to Levi's Authorship or Age.

All three second edition mss. share the features above, distinct from the first edition mss.

**Moscow 30 (Mo30)** Contains all second edition problems.  
Complete second edition. Similar to but much less reliable than Mo1063. Bound with a collection of other mathematical works copied by the same scribe, Gad Ashtrok. No margin notes. Sephardic script with Italian elements. Dated 1503.

**Jerusalem (Je)** Contains all second edition problems.  
Complete second edition. Reliable version. Very few margin notes. Space is left for all figures, but none are filled in. Sephardic script, possibly Provençal. Dated 1410.

**Moscow 1063 (Mo1063)** Contains all second edition problems.  
Complete second edition. It includes many margin notes including the insertion of much missing text from the first edition. This makes it perhaps the most reliable of all the mss. Similar to Mo30. Provençal Script. Estimated date – late 14<sup>th</sup> / early 15<sup>th</sup> century.

### **Stemma of the Twelve Manuscripts of *Maaseh Hoshev***

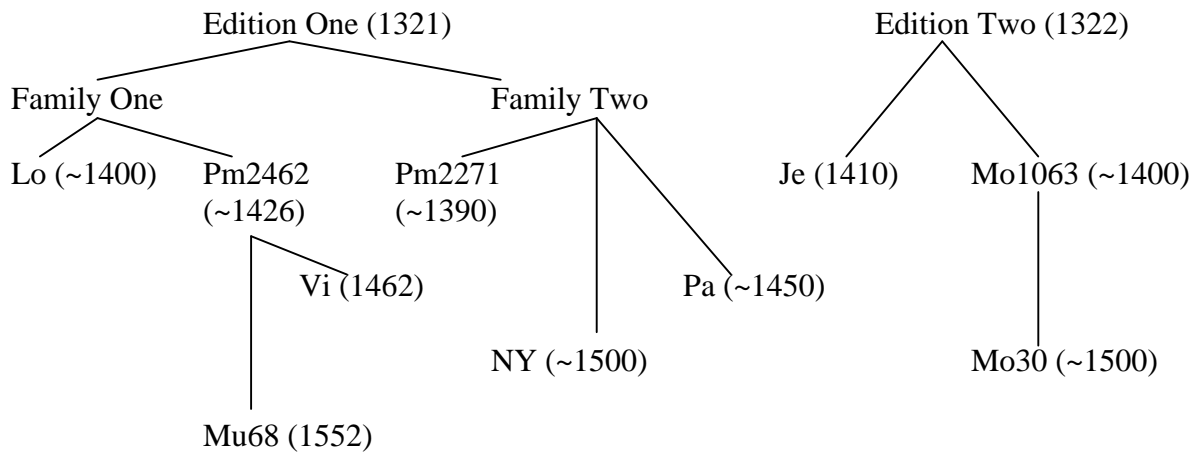
A stemma based on all the information above, is shown below. The dates of the mss. are shown, (some are approximate, indicated by ~), and correspond to the depth in the tree, where each level is about 50 years. None of the mss. represents an autograph, and none are completely error free. The two editions are marked on separate trees. In edition one, there are two families of mss. The first family contains the Vi, Pm2462, Mu68 and Lo mss. while the second family contains the Pm2271, Pa and NY mss. The problem section is completely missing from Va and omitted in Mu36, hence for the purposes of this study they are left out of the stemma. A more complete stemma must wait until the publication of a new critical edition of the whole book, which would include the portions in Va and Mu36.

The two families of edition one are identified by dozens of similar errors, omissions and particular choices of phrases. Within the first family, Vi, Pm2462 and Mu68 are particularly similar throughout. Lo shares most of the features of the family but there are many places where it is correct and the other members of the family have errors. Furthermore, Lo has missing

phrases which appear in the other members of the family. Hence it is likely that Lo represents a parallel branch down from the archetype of family one edition one. On the other hand, the other three mss. in the family are virtually identical except for very minor differences most of which are missing phrases in Vi which appear in Pm2462 and Mu68. Vi has a few pairs of letters that are extremely hard to distinguish throughout the ms., making it very unlikely that it was copied reliably, yet Pm2462 and Mu68 do not have any errors in these letters. Finally there are no phrases missing from Pm2462 that appear in Vi or Mu68. Hence the stemma indicates that Pm2462 descends from the family one archetype, and both Vi and Mu68 descend from it. This is also consistent with the dates of these mss.

In the second family, Pm2271, Pa and NY share dozens of similar minor errors and variations which are distinct from family one. The most striking example is that both Pm2271 and Pa have a small fragment of problem 14 edition two, which appears just before problem 18. NY is missing all the pages from the middle of problem 17 on, so it is impossible to check that this striking feature is part of NY, nevertheless NY does share the other similar features of family two. It should be noted however, that NY is the most enigmatic of all the mss. It is not faithful to the precise wording of the original, substituting synonyms and alternate grammatical forms throughout. Finally, both Pa and Pm2271 have missing phrases not found in the other ms. implying that neither is the likely source of the other. Hence the stemma indicates all three descending from the archetype family two.

In edition two, Mo30 is similar to Mo1063 but has many more errors and omissions. Je, in comparison, has a number of minor differences and one major difference. It includes two paragraphs in problem three which are absent in Mo1063 and Mo30. Hence the stemma indicates Mo30 descending from Mo1063 which in turn descends from the edition two autograph. Je descends in parallel from the edition two autograph. Of course this is consistent with the dates of these mss. Finally, it should be noted that both Je and Mo1063 are very reliable. In particular, Mo1063 has a large number of margin comments including many corrections and the insertion of missing sections, many from edition one. These insertions include the above mentioned two paragraphs present in problem three of Je.



#### IV. An Overview of the Missing Problems.

The mathematical theme underlying all of the missing problems is applications of proportions. Although, the mathematical foundations are based on Euclid, the problems and techniques themselves range from the trivial to the complex. The easier problems appear at the start and they get progressively harder. The variety of illustrations include:

- a. Linear and quadratic equations with one unknown.
- b. A variation of the well known Cistern problem.
- c. Least common multiples and weighted average.
- d. Various simultaneous linear equations with two or more unknowns.
- e. Various business related problems of purchase, sale and profit.

The format of each problem is a general question, followed by a general method of solution, followed by an example, followed by an explanation or proof that often refers back to material earlier in the book. Sometimes the explanation is left out or shortened, and sometimes it is drawn out into a formal style proof. Sometimes there is more than one example, and sometimes variations or generalizations are discussed. The overall format, however, is consistent throughout. When variations of the problem are discussed or a digression ensues, then for the purpose of numbering the problems, I count a new problem only when it begins with the word *sh-ela* (question or problem).

#### V. Historical Context of the Problems

##### Levi's Sources

Levi presumably did not read Latin or Arabic, yet he certainly had access to Hebrew translations of Euclid, and probably had access to translations of other sources. There are

clear explicit and implicit references to Euclid throughout *Maaseh Hoshev*, a few appearing in the problem section.

The connection of medieval Hebrew scholars with the classical Greek works of Aristotle and Euclid is well known. What is not well known is to what extent Levi (and other medieval Hebrew scholars) had access to Latin and Arabic sources, and to what extent, if any, these sources influenced him. A comprehensive comparison of Levi's problems with other Latin and Arabic collections of problems is unquestionably important. I begin closer to home, with a comparison of a similar Hebrew book by Abraham ibn Ezra. Abraham ibn Ezra (1090-1167) was a Spanish Jewish philosopher, poet, grammarian and mathematician.

Outside of Euclid, *Masseh Hoshev* contains not the slimmest shred of an explicit or implicit reference to any other source. Nevertheless, it is interesting to note the following similarities to Abraham ibn Ezra's *Sefer Hamispar*. *Sefer Hamispar* (1146), is a book on arithmetic and calculation in seven sections with a short introduction explaining the decimal system and the use of 0 as a place keeper. (Note that both Ibn Ezra and Levi use the decimal system for integers but base 60 for fractions, as was common in that day.)

Section One -	Multiplication.
Section Two -	Division.
Section Three -	Sums.
Section Four -	Differences.
Section Five -	Fractions.
Section Six -	Ratios.
Section Seven -	Square roots.

In content, the only thing in *Maaseh Hoshev* which is completely absent in *Sefer Hamispar*, is the material on combinatorics and series.

The sixth section on ratios is in style and content similar to Levi's section on proportions but not as rigorous and not as long. It also has a list of problems, which Ibn Ezra says are necessary to flush out the full power of his methods. Unlike Levi who provided careful proofs, general examples and special cases, Ibn Ezra's style is to explain his ideas through numerous specific examples, relying on quantity to substitute for rigor. Noteworthy in both books, in contrast to most Arabic works on arithmetic and algebra, is the absence of geometric proofs and arguments.

Although Levi makes no reference to Ibn Ezra's work, the similarities in organization and content are striking. This is especially true in the problems which appear in the sections on ratios and proportions. They both start with simple linear equations and continue with practical economic problems, such as people hiring other people to do a certain job and the latter did only a partial job etc. There are of course some major differences too. Levi has 21 problems in two editions and Ibn Ezra has only 14. Levi's problems are more rigorously presented, and his latter problems deal with more complicated simultaneous linear equations that Ibn Ezra does not approach. In general, Levi has more difficult problems in his list, and has more innovative approaches to their solutions. This is consistent with the rest of his book

which surpasses Ibn Ezra in ingenuity and creativity. A final difference is that all of Levi's problems are secular in nature, while Ibn Ezra includes a problem about inheritance that is religious in nature. This latter point is interesting, implying that Levi may have intended his work for a wider audience.

Levi probably had access to *Sefer Hamispar*, and because it was written in Hebrew, Levi would have been able to read it without difficulty. Nevertheless, despite the similarities in structure between *Sefer Hamispar* and *Masseh Hoshev*, one must be careful not to jump to conclusions, because there is no firm evidence or reference to back up such speculation.

Ibn Ezra, was fluent in Arabic, and therefore had access to Arabic sources without the need of translation. In particular, many of Ibn Ezra's mathematics problems are in the context of business transactions, whose study in Arabic is called *mu'amalat*. There is a work by Ibn al-Haytham (11<sup>th</sup> century) [28] on *mu'amalat* which was certainly accessible and might have been available to Ibn Ezra. This work includes problems which require the manipulation of fractions and whose themes are seen in the early problems of Levi's list.

Hence Levi could have been influenced by the Arabic tradition indirectly, through Ibn Ezra. However, Levi's later problems, and certainly the rest of *Maaseh Hoshev*, go far beyond these basic themes, so this possible influence is unlikely to have been too significant.

There are other possible sources, both Latin and Arabic, that must be considered. These sources might have been available to Levi in translation, or indirectly through the work of other Hebrew scholars fluent in Arabic or Latin. However, any conclusions about a definitive source in this area are even more speculative than the connections with Ibn Ezra.

One possible Latin source is *De numeris datis* by Jordanus de Nemore [14]. Jordanus flourished in late 12<sup>th</sup> and early 13<sup>th</sup> century France, and his book is a compendium of about 100 algebra problems of linear, quadratic and (only one) cubic equations with general solutions, proofs and examples, divided into four sections. For example, the third problem in section one of his book is to solve the system of equations  $x + y = a$ ,  $xy = b$ . The second section of the work is on problems described through proportions, exactly the kind of problems that Levi treats in his list of 21. Despite the fact that Jordanus' work is quite comprehensive, and that there are certainly similarities between his problems and Levi's, almost none of the problems in Levi's list correspond *directly* to any one of the hundred in Jordanus. Here is an example that comes very close. Compare Jordanus' problem 12a from section two, with Levi's problem 20.

$$\text{Jordanus II-12a: } (x+a)/(y-b) = c, (y+d)/(x-e) = f.$$

$$\text{Levi 18: } (x+y)/(z-x) = a, (x+z)/(y-x) = b.$$

Levi's problem is indeed a special case of Jordanus'. This can be seen by setting  $x=y$ ,  $a=x$ ,  $c=a$ ,  $y=z$ ,  $b=x$ ,  $d=x$ , and  $e=x$ . Levi's proof and discussion of this problem is also lengthier and more convoluted than Jordanus'. It is not likely that Levi would have taken a source with a more general problem and solution, and presented only a special case of the problem with a more difficult proof. Furthermore, Jordanus presents all his problems directly

(i.e. the words describe the equation) while Levi presents most of his problems hidden in the disguise of a business problem. The few problems in Levi that are presented directly (problems 16-21) cannot be matched exactly with any of the problems in Jordanus' list, see the example above.

Even if there was a translation of Jordanus' work into Hebrew, and there is no evidence for such a translation, it seems unlikely that Levi studied the work. Nevertheless, it is worth noting that both authors present their work in a rigorous way, with general statements and proofs supported by specific examples, and both make use of the idea of reducing one equation to a canonical form after which it is solved using the solution to the canonical form. It should also be noted that Jordanus has a larger more comprehensive work, *De elementis arithmetice artis*, which became the standard source for theoretical arithmetic in the middle ages [4]. The work is broader than *Maaseh Hoshev* and includes much of the content from the list of theorems in part 1 of *Maaseh Hoshev* except perhaps the combinatorial theorems, but there is nothing in it like the problem section presented here.

It is well known [7; 13] that Levi studied the scientific works of Ibn Rushd (in Hebrew translations), wrote supercommentary on his works, and taught students Aristotelian science. Hence one would expect that, if possible, he would have used similar sources for his mathematical work. There were Arabic works on mathematics translated to Hebrew, but some, such as the Algebra of Abu Kamil (850-930) [22], were translated after Levi died, (by Mordecai Finzi, 15<sup>th</sup> century, in the case of Abu Kamil). Other later works, such as the Algebra of Omar Khayyam (1044-1123) [15], are unlikely to have been sources because they focus much more on various categories of quadratic and cubic equations, than on linear equations. Levi's work in the problem section goes deeply into various linear and simultaneous linear equations, but hardly at all into higher order equations. Also, the later Arabic works have many geometric proofs and arguments completely absent in Levi's work. Levi's proofs are longwinded but contain innovative arguments which are strictly algebraic. I argue in [21] that one of Levi's contributions in the development of medieval mathematics is precisely this shift from a geometric view of algebra to a combinatorial view.

There is one work on arithmetic by al-Hassar (13<sup>th</sup> century), which was translated into Hebrew as *Sefer ha-Heshbon*, but it is unpublished and only extant in three manuscripts (Moscow, Oxford, and Vatican). The first two mss. indicate that the translator was Moses ibn Tibon who translated the work in Montpellier in 1271. The third (Vatican) has an anonymous translator. If Levi had used an Arabic source, this would fit the bill. It was in the right place at the right time, available in Hebrew, and the content focussed more on arithmetic and proportions than on higher order algebra.

*Sefer ha-Heshbon* by al-Hassar is a large work on arithmetic of integers and fractions, divided into many sections and subsections. The three extant manuscripts of the Hebrew translation differ greatly especially in the sections at the end. It deserves a separate study, but here we will present a brief overview in context of its possible connection to *Maaseh Hoshev*. In the book, al-Hassar discusses addition, subtraction multiplication, division, square and cube roots of both positive integers and fractions. He gives general methods followed by numerous particular examples to make the method clear. He gives no rigorous proofs in the

book, which reads more like a practical manual of calculation than anything else. This is in great contrast to Levi's rigorous Euclidean style. Furthermore, although al-Hassar has literally hundreds of problems and examples, hardly any are presented in the puzzle style that Levi uses for his problems.

Nevertheless, there are still similarities between the two works. First of all both books deal with similar subject matter. This is itself is not so surprising since arithmetic is such a basic and practical topic. However, *Maaseh Hoshev* contains sections on sums and combinatorics whose rigorous presentation is absent from most early medieval works on arithmetic and algebra. Although *Sefer ha-Heshbon* by al-Hassar contains no combinatorics, it does contain a whole section on sums. In particular, it gives the formulae for sums of consecutive integers, consecutive odd integers, consecutive squares and consecutive cubes. All these formulae were known well before al-Hassar's time, and some can be found in other Arabic works like those of al-Karaji (1000) and al-Samaw'al (1125-1180), but al-Hassar is later than these, and his work has a Hebrew translation. Moreover, although al-Hassar does not rigorously prove his formulae like Levi does, he does give constructive examples from which proofs could easily be derived.

Overall, *Maaseh Hoshev* is broader in scope, more rigorous, and more ingenious than *Sefer ha-Heshbon*. However the latter could have been a source for the former, especially for the sections on square roots, cube roots and sums whose details are not found in Euclid. Unfortunately, as far as our problem section goes, there is little in al-Hassar's work that could have served as source material for Levi.

### **Who Studied *Maaseh Hoshev*?**

Although Glasner [13] argues convincingly for a school of Levi, and identifies possible students, there is no direct evidence that there were students of *Maaseh Hoshev*. I argued earlier of the possibility of students being involved in the writing of the second edition, but the only direct evidence for this is the omission of Levi's name from the second edition colophon. Despite the existence of twelve extant mss., it is not likely that many people studied and understood everything in *Maaseh Hoshev*. In every ms., the kinds of errors that appear imply that the scribes themselves were hired copyists, rather than students. The errors show no understanding of the underlying content, and often render a section completely unreadable. Without access to multiple copies of the work, these sections are not possible to reconstruct. Problem 17, in particular, which appears only in the first edition, is so badly mangled in every ms., that no one in the last 600 years was likely able to work through all the details.

On the other hand, there were at least a few serious readers of the text. I note that in Mo1063, a second edition ms., there are extensive comments in the margins. These comments include a great deal of the missing material from the first edition, as well as context sensitive corrections. This implies that the text was studied and cross-referenced with access to mss. of both editions. The Lo ms. also has extensive corrections in the

margins, some by Mordecai Finzi, implying a careful study. Nevertheless, it should be noted that there are no direct known references to *Maaseh Hoshev* in the literature until 1909.

## **VI. Introduction to the Translation**

I attempt to balance faithfulness to the author's language and style, with readability. Despite Levi's genius, his style is tedious, repetitive and ponderous. I do not wish to burden the reader, but nor do I wish to hide the natural flow of his style.

I have freely added punctuation for readability. Levi has virtually no punctuation nor does he number his problems as he does his theorems in part one. The numbering of the problems is ours, and serves as an index for referencing. Commentary is added, to clarify points about content, language, translation, and cultural references. To make it easiest for the reader, I placed the commentary directly under the section being discussed, rather than in a separate section with cross references. The commentary is distinguishable from the translation, by the indentation and the smaller size font. The problems that are specific to one edition but not the other, are marked **Ed. 1** or **Ed. 2** at the start of the problem.

### **Errors and Corrections**

All errors are discussed in the commentary. It is not always easy to tell when an error is the result of the scribe or whether it is original with Levi, although generally the errors are the former type. The commentary distinguishes between these two kinds of errors, and attempts whenever possible to suggest plausible explanations for the origin of the error. When the mss. all agree on an error, I leave the error in the text, using the commentary to discuss alternative corrections. When each ms. presents a different incorrect version, I reconstruct the best correct version in the text itself, leaving the critical apparatus as a record of the errors.

### **Standard Hebrew Numbering**

With regard to Hebrew numbering, some explanation is necessary. The standard Hebrew numbering scheme, still in use today for many purposes, is a mix of positional and symbolic styles using the 22 letters of the Hebrew alphabet. The first 9 letters of the alphabet indicate the numbers one through nine. The next nine letters indicate the numbers 10 through 90. The next four indicate the numbers 100 through 400. Any number up to 499, is uniquely represented by using the appropriate letters for each base ten digit. A zero digit is indicated by having no letter at all. There are some exceptions to this, one of which is followed by Levi, and that is not to use the tenth letter followed by the fifth letter for 15, because this spells the name of God. Instead he uses the ninth letter followed by the sixth. For example, 115 is the 19<sup>th</sup> letter (value 100), followed by the ninth and the sixth; 207 is the 20<sup>th</sup> letter (value 200), followed by the seventh. When the need comes for numbers larger than 499, the largest possible valued letters are used. For example, 790 would be the 22<sup>nd</sup> letter (value 400), followed by the 21<sup>st</sup> letter (value 300), followed by the 18<sup>th</sup> letter (value 90); and 850 would be the 22<sup>nd</sup> letter (value 400), followed by the 22<sup>nd</sup> letter (value 400), followed by the



14<sup>th</sup> letter (value 50). When really big numbers are needed, and multiples of 400 do not suffice, the words for “thousand” are combined with the numbering system to get anything needed. For example, 32 million, would be written as the 12<sup>th</sup> letter (value 30) followed by the 2<sup>nd</sup> letter followed by “thousand thousand”.

Note that many numbers look like Hebrew words. The example of 15 and God has already been noted, but this is just one of many common words. What’s more, single Hebrew letters act as prepositions or articles when added to the front of a word. These include: and, with, on, to, in, from, etc. Hence it is ambiguous when these letters appear in front of numbers that happen to be words. One might not know whether it is “from” 8 you are subtracting, or whether it is 48 (the 4<sup>th</sup> letter being the preposition “from”). In order to help a reader resolve this ambiguity, small slashes or dots are made over the letters that are being used as numbers, and of course the context helps.

### **Levi’s Notation for Numbers and Variables**

Levi uses two numbering schemes for integers. One is the standard modern system, with the first nine letters of the Hebrew alphabet for the digits 1-9, and a circle for zero, and the other is the standard Hebrew system discussed above. It is not hard to differentiate what system is being used and he mixes them freely.

However, he also uses Hebrew letters as variables. He even uses sequences of Hebrew letters, to denote either the product or sum of the variables, as the context demands. Since so many more combinations of letters are allowed for variables, this exaggerates the ambiguity between variables and words. The difficulty of distinguishing leading letters that are articles or prepositions is also increased. In all these cases, the slashes are used, but it is not always easy to tell when a slash is there and when it is not. To distinguish between a sequence of variables that is a product from one which is a sum, he usually precedes the variables with the word *murkav* (composite) when it is a product, and *misparei* (the numbers) when it is a sum. However, he is not always consistent about this. Levi also uses a sequence of two variables to denote one number, in the geometric sense of a line segment. When doing so, he precedes the two variables with the word *mispar* (number).

Once, in Problem 17, Levi runs out of variable names. The 22 letters of the Hebrew alphabet do not meet his needs, so he resorts to using five extra symbols. These *sofit* symbols are those used for the Hebrew letters that appear differently at the end of a word from the way they do in the middle of a word, resulting in 27 distinct symbols. This was no doubt the cause of confusion and corruption in the copying of Problem 17. The scribe could err in many new ways. Not only could he forget a slash, or omit a space, but he could write a specific letter either in its ending form, or in its normal form. This ambiguity confused the scribe who did not likely appreciate the serious error of substituting an end form for a normal form, of a letter that happens to appear at the end of a list of variables. Nor did the scribe appreciate the similar serious error of substituting a normal form for an end form, of a letter that happens to appear in the middle of a list of variables.

Levi writes out fractions in long hand, as in “2 of 9 parts of a dinar”, when describing fractions of things, but he also writes fractions using base 60 with the standard Hebrew system to write the numbers from 1 to 59. Spaces are left between the successive base 60 positional values. This introduces more possibilities for error, as the scribe must be careful in the spacing, so as to distinguish between 50; 9, and 59, which are identical in the standard Hebrew system except for the space in between the 15<sup>th</sup> letter and the 9<sup>th</sup>.

### **Translation of Levi’s Notation**

I translate numbers to equivalent modern digits when they are written as numbers either in the Hebrew or modern style, but use words when the numbers are written out as words. I translate variables using letters, using the *n*th letter in the English alphabet for the *n*th letter in the Hebrew alphabet. When a string of letters appears that means sum, literally “the numbers ABC”, then I translate it as “the number A+B+C”. When the string of letters means product, I use “the product ABC” literally as it is written.

## **VII. Translation of Problems 1-15 from *Maaseh Hoshev* with Commentary**

We set forth for you a variety of problems, so that you may understand similar ones.

**1. A Problem is Posed:** We extract a given fraction or fractions from an unknown number and the result is a given number. What is the unknown number?

The method here is to take the prime denominator of all the fractions, to extract these fractions from this denominator, and make a note of the result. Multiply this denominator by the second given number and divide the result by the noted value, to get the requested number.

Levi defines, *moreh rishon*, or prime denominator, in section five of part two as the smallest integer that measures the denominators of the fractions; that is, the least common denominator. He often uses just the word *moreh*, which means simply denominator, in context, to denote the prime denominator or least common denominator. When he does this, I translate it as common denominator.

For example, 2 fifths plus 3 fourths plus one third of a number equals twenty, and we want to know the value of the whole number. The prime denominator of these fractions is 60. We extract these fractions from it, to get 89, and this is noted. Multiply the common denominator by twenty to get 12 hundred, divide this by the noted value, to get 13 whole and 43 of 89 parts of the whole, and that is what was requested. You can check this if you wish.

Levi’s example in modern notation:  $(2/5 + 3/4 + 1/3)x = 20$ .

He directs us to combine the fractions to get 89/60, and says that  $x = 20 (60/89) = 13 + 43/89$ .

He explains below in great detail that 89 is to 60 as 20 is to *x*.

This is right, because the ratio of 2 fifths of 60, to 60, equals the ratio of 2 fifths of the unknown number, to the unknown number. Similarly, the ratio of 3 fourths of 60, to 60, equals the ratio of 3 fourths of the unknown number, to the unknown number; and the ratio

of a third of 60, to 60, equals the ratio of a third of the unknown number, to the unknown number. When we add them together, the ratio of all the fractions taken from 60, to 60, equals the ratio of these all the fractions taken from the unknown number, to the unknown number. Hence, the ratio of 89 to 60 equals the ratio of 20 to the unknown number. Hence, the product of 60, which is the second, with twenty, which is the third, equals the product of 89, which is the first, with the unknown number. And use this as a model.

The references here to “first”, “second” and “third” come from Section f of part two where Levi reminds us from Euclid, that four numbers are related if the first times the fourth equals the second times the third. That is  $A/B = C/D$  if and only if  $AD = BC$  (Book 7 Proposition 19 of The Elements).

**2. Problem:** Certain fractions of an unknown number exceed a second set of fractions of this unknown number by a given amount, what is the whole number?

Take the common denominator of all these fractions, extract from it the first set of given fractions and make a note of the result. Also extract from it the second set of given fractions and make a note of this second value. Subtract the second noted value from the first, and the remainder is the adjusted noted value. Multiply the common denominator by the given amount, divide the result by the adjusted noted value, and the result is the requested number.

For example, 3 sevenths plus 4 fifths of an unknown number exceed 2 thirds plus a fourth of the unknown number by twenty, and we want to know the value of this number. The common denominator of all these fractions is 420, and 3 sevenths plus 4 fifths of it is 516. And 2 thirds of 420 plus a fourth of it is 385. The first set of fractions exceeds the second set of fractions by 131. We multiply the common denominator by twenty and divide the result by 131, to get 64 whole and 17 of 131 parts of the whole. This is what was requested, and you can check this if you wish.

Levi’s example in modern notation:  $(3/7 + 4/5)x - (2/3 + 1/4)x = 20$ . The “proof” below is tedious, methodical and correct. Of course, he does not use any of the algebraic tools routinely expected of today’s grade school students, substituting in their place an argument using proportions.

This is right, because the ratio of the sum of the first set of fractions extracted from 420, to 420, equals the ratio of the first set of fractions extracted from the unknown number, to the unknown number. By exchanging them, the ratio of the first set of fractions extracted from 420, to the first set of fractions extracted from the unknown number, equals the ratio of 420 to the unknown number. Similarly, the ratio of the second set of fractions extracted from 420, to the ratio of the second set of fractions extracted from the unknown number, equals the ratio of 420 to the unknown number. By exchanging them, the ratio of the first set of fractions extracted from 420, to the second set of fractions extracted from 420, equals the ratio of the first set of fractions extracted from the unknown number, to the second set of fractions extracted from the unknown number.

By separating, the ratio of the first set of fractions extracted from 420, to its excess over the second set of fractions extracted from 420, which is 131, equals the ratio of the first set of fractions extracted from the unknown number, to twenty, which is its excess over the

second set of fractions extracted from the unknown number. By exchanging them, the ratio of the first set of fractions extracted from 420, to the first set of fractions extracted from the unknown number, equals the ratio of 131 to twenty. But we already showed that the ratio of the first set of fractions extracted from 420, to the first set of fractions extracted from the unknown number, equals the ratio of 420 to the unknown number. Hence, the ratio of 420 to the unknown number, equals the ratio of 131 to twenty. Hence, the product of 420 with twenty equals the product of 131 with the unknown number. And use this as a model.

**3. Problem:** The cost of a given quantity of a product is given. What is the cost of a second given quantity of this product?

Multiply the given cost by the second given quantity, divide by the first given quantity and this is what was requested.

For example, the cost of 11 measures of wheat is 7 dinars, and you want to know the cost of 15 measures of wheat. Multiply 7 by 15 and divide by 11, to get 9 whole and 6 of 11 parts of one, which is what was requested. That is, the cost of 15 measures is 9 dinars and 6 of 11 parts of a dinar.

This is right, because the ratio of the first quantity to the second quantity equals the ratio of the known cost to the unknown cost. This is self evident. Thus the product of the second quantity with the known cost equals the product of the first quantity with the unknown cost. And use this as a model.

Here is the method to convert a given fraction of the day into hours and fractions of an hour, or to convert a given fraction of the litra, which is 20 dinars, into dinars and peshuts and fractions of a peshut, or to convert anything that is made up of a known number of parts.

For example, we want to know how many hours and fractions of an hour there are in 83 of 109 parts of a day. We know that the number of hours in a day is 24. So the ratio of 83 to 109 equals the ratio of the unknown number of hours to 24. Multiply 24 by 83, divide by 109, and that is what was requested. That is approximately 18 hours, 51 firsts and 12 seconds.

Levi uses base 10 for integer parts and base 60 for fractional parts of a number. This was very common in his time. Therefore, “firsts” and “seconds” should be interpreted appropriately here as minutes and seconds. Note however, that the answer is wrong. The correct answer is approximately 18; 16, 31. In other examples of conversion to base 60 throughout these problems, Levi gets the answers precisely correct, so one should not speculate that there is some flaw in his method, rather that the numbers were corrupted, and represent scribal errors. This is the first error of several similar errors found in all the extant mss.

If the example used a litra, then since we know that the number of dinars in a litra is twenty, the ratio of 83 to 109 equals the ratio of the unknown to twenty. Multiply twenty by 83 and divide by 109, to get 15 dinars and 15 of 109 parts of a dinar. You can now find the value in peshuts by multiplying 12 by 106 and dividing by 109, to get 11 peshuts and 73 of

109 parts of a peshut, which is approximately 2 thirds of one peshut. In this way, you can handle similar things.

The passage above states explicitly that there are 20 dinars in a litra and implies that there are 12 peshuts in a dinar. A *litra* is a measure of weight, translated commonly as a pound, and the peshut and dinar are coins, hence there are 240 peshuts to a pound.

The *peshut* is almost surely the *denier*, a paper thin lightweight coin of mixed silver and copper, and the standard coin of Europe for 500 years between the 8<sup>th</sup> and 13<sup>th</sup> centuries. The dinar is just as surely the *gros tournois*, a heavy royal French silver coin introduced by king Louis IX of France in 1266, which became the standard coin of Europe from the late 13<sup>th</sup>-14<sup>th</sup> century [19]. The Jewish traveler Eshtori ha-Farhi mentions the use of coins in Provence in the 14<sup>th</sup> century. Farhi writes that there is a small coin that the Arabs called a *pashit*, twelve of which are one "white tournois" [25]. No doubt that the Arab *pashit* is the Hebrew peshut, and this in turn is the denier, while the white tournois is the gros tournois. There were indeed 12 deniers in a gros tournois and 240 deniers in a pound [34; 35]. Note the 12 crowns around the circumference of the coin in the figure below.

<Insert Coin picture here>

**Figure 3. The gros tournois. Note the 12 markings around the circumference.**

The terms dinar and peshut are found in the Talmud, referring to generic large and small denominations respectively. Levi seems to use the anachronistic Talmudic names for real coins of his day. Note that what was called a dinar in his day, was the standard muslim gold coin, worth a great deal more than Levi's dinar. Later on, Levi will give a problem where he talks about dinars and gold dinars. Presumably, the latter is the dinar of his day, and the former is the gros tournois.

The next four paragraphs are found in all the first edition mss. In the second edition, they are present in the Jerusalem ms., copied in the margin of Moscow 1063, and omitted from Moscow 30.

Here is the method for converting a variety of fractions to one fraction. Take the prime denominator of the various denominators of the fractions, extract the fractions from it, and the result is the fraction of the common denominator.

For example, if you wanted to convert 3 sevenths and 4 parts of 17 to a single fraction, then take these parts from the common denominator which is 119, and the result is 79 parts of 119.

Here is the method for converting the parts of a given fraction to a unit fraction. Divide the denominator of this fraction by the numerator, and the result is what was requested.

Levi uses these last two methods throughout the problem section.

For example, if you wanted to convert 3 sevenths to a unit fraction, divide seven by 3, to get 2 and a third. This is one of 2 and a third parts of that from which we wanted to extract 3 sevenths. And use this as a model.

Note “This” in the second sentence of the last paragraph refers to  $3/7$ . The wording in Hebrew is also ambiguous.

**4. Problem:** A traveler with a uniform movement, walks a fixed distance in a given amount of time. How far will he walk in a second given amount of time?

Multiply the second given amount of time by the fixed distance, divide the result by the given amount of time, and you will get what was requested. If one or both of the times contained hours, then convert the amount of time to hours, where hours are the whole unit.

For example, the traveler traveled 7 measures, 36 firsts and 57 seconds in 13 days, and we want to know: how far would he travel in 3 days, 17 hours, 52 firsts and 16 seconds? We convert the days to hours so that the first given time is 312 and the second time is 89 whole, 52 firsts and 16 seconds. We multiply 7 whole, 36 firsts and 57 seconds by 89 whole, 52 firsts and 16 seconds. We divide the result by 312 to get 2 whole, 10 firsts and 56 seconds. This is the number of measures that he walked in the second given amount of time.

This is an error. The correct answer is approximately 2; 11, 37. Note that the number 52 and 22 look alike in Hebrew, and that if 22 is used instead, then the answer is 2; 10, 53, which is much closer to the given answer but still incorrect. The Pm2271 ms. has 22 for the first occurrence of 52, but not for the second. It is not clear what the correct numbers should be in this example.

This is right, because in a uniform movement, equal times make equal measures. Therefore, the ratio of time to time equals the ratio of journey to journey.

You can now consider the reverse problem. That is, the problem is posed given a uniform movement where a given distance is walked in a given amount of time. How much time will it take to walk a second given distance? Multiply the second given distance by the given time, converting the days to hours to make it easier. Divide the result by the first given distance and this is what was requested.

For example, the traveler walks 7 measures, 36 firsts and 57 seconds in 13 days. We want to know: how much time will it take him to walk 3 measures? We multiply 3 measures by 312 whole, and divide by 7 whole, 36 firsts and 57 seconds, to get 122 hours, 54 firsts and 7 seconds, which is the unknown time. This is explained by the very same reason as before.

This answer is correct to the nearest second. Note that 54 firsts is 54 minutes.

**5. Problem:** If two travelers, one faster than the other, are traveling with uniform movements, and the distance between the faster, who is behind, and the slower, is a fixed measure; how much time will it take the faster to catch the slower?

Divide the fixed measure, by the excess distance traveled by the faster over the slower in an hour. The result is the number of hours and fractions of an hour that it takes the faster to catch the slower.

For example, the faster travels 2 measures and 37 seconds in an hour, the slower travels 30 firsts and 24 seconds in an hour, and the distance between the slower, who is in front, and the faster, is 29 measures and 45 firsts. The excess distance traveled by the faster over the slower in an hour, is one measure, 30 firsts and 13 seconds. We divide this into the given distance to get 19 hours, 47 firsts and 9 seconds. The reason for this is clear from the previous discussion.

The result 19 hours, 47 minutes and 9 seconds is correct to the nearest 360<sup>th</sup> but not exact. In this case, Levi usually says that the result is a close approximation, but here he does not.

**6. Problem:** A certain full container has various holes in it. One of the holes lets the contents of the container drain out in a given time; a second hole lets the contents drain in a second given time; and so on for each of the holes. All the holes are opened together. How much time will it take to empty the container?

This is a famous recreational mathematics problem, variations of which have appeared from ancient times until today [33].

First, calculate what drains from each hole in one hour and add the values all together. Note the ratio of this to the full container. This ratio equals the ratio of one hour to the time needed to empty the container.

For example, a barrel has various holes: the first hole empties the full barrel in 3 days; the second hole empties the full barrel in 5 days; another hole empties the full barrel in 20 hours; and another hole empties the full barrel empties in 12 hours. Therefore, the first hole empties one of 72 parts of the barrel in an hour; the second hole, one of 120 parts; the third hole, one of 20 parts; and the fourth hole, one of 12 parts. When we add them all up, the total that empties from all the holes in an hour is 56 of 360 parts of the full barrel. We divide 360 by 56, to get 6 whole and 25 firsts and 43 seconds. Therefore, the time to empty the barrel is approximately 6 hours, 25 firsts and 43 seconds. The reason for this is clear.

Here Levi implicitly uses the method of converting a fraction to a unit fraction, as discussed in Problem 3. Once again note that “firsts” corresponds to minutes and “seconds” corresponds to real seconds.

**7. Problem:** The cost of a given item is a certain number of dinars. What is the cost of a given set of fractions of this item?

Take the common denominator of all the fractions, and extract the fractions from it. Then find the cost of this result, multiply by the number of the item, and divide by the common denominator, to get what was requested.

For example, the cost of a gold dinar is 25 dinars, and we want to know: what is the cost of a dinar with half of it, 2 sevenths of it, and 3 fourths of it? The common denominator of these fractions is 28. You extract these fractions from it to get 71. You multiply 71 by 25 to get a thousand and 775, and this is the cost of these extracted fractions. Multiply this number, by the size of the purchase which is one, and divide by 28, to get 63 whole and 11 of 28 parts of one, which is the cost. That is, 63 dinars and 11 of 28 parts of a dinar. You can check this if you wish.

Here we see that Levi distinguishes between the gold dinar which is the standard Muslim dinar, and the dinar which is the gros tournois. See the commentary in Problem 3.

This is right, because the ratio of the fractions extracted from the common denominator, to the fractions extracted from the gold dinar, equals the ratio of the common denominator, to one, which is the size of the item. Thus the ratio of the cost of the fractions extracted from the common denominator, to the cost of the fractions extracted from the gold dinar, equals the ratio of the common denominator to one, because it is clear that the ratio of item to item equals the ratio of cost to cost. Hence, the product of the cost of the extracted fractions from the common denominator, with one, equals the product of the cost of the extracted fractions from the dinar, with the common denominator. And use this as a model.

The explanation is one and the same if the size of the item being sold is arbitrary. For example, the cost of 7 measures of wheat is 25 dinars, and we want to know: what is the cost of the wheat with half of it, and 2 sevenths of it, and 3 fourths of it? We extract the fractions from the common denominator which is 28, to get 71. The cost of these is 253 dinars and 4 sevenths of a dinar. Multiply this by the size of the item being sold, which is 7, divide by the common denominator, which is 28, to get 63 dinars and 11 of 28 parts of a dinar. And use this as a model.

**8. Problem:** A merchant sells items of different costs, and a buyer wants to purchase the same measure of each item using a given quantity of his money.

As was the standard practice, Levi describes the general case by using a particular example without loss of generality. He does this throughout the book.

This problem is the general case of: Given  $a, b, c, d$  and  $M$ , find  $x$  such that  $ax + bx + cx + dx = M$ .

The method is to add up the costs per measure of each product and note the result. From each item, he should take the fraction of a measure, equal to the ratio of the total money in his hand, to the noted result. This is what was requested.

For example, a merchant is selling four drugs. The cost of the first drug is 7 peshuts per litra; the cost of the second drug is 8 peshuts per litra; the cost of the third drug is 10 peshuts per litra; the cost of the fourth drug is 15 peshuts per litra. The buyer comes with 3 dinars to buy the same weight of each. With 3 dinars and 4 peshuts he can get a litra of each one, and the ratio of 3 dinars, to 3 dinars and 4 peshuts, is 9 tenths. Therefore, he should get this much from each drug; that is, 9 tenths of a litra. The total cost is 3 dinars, and you can check this if you wish.



This problem confirms the exchange rate between the peshut and the dinar, namely that there are twelve peshuts in a dinar. This information helps identify these coins. See commentary in Problem 3.

This is right, because the ratio of what he takes from each of the litras, to the litra, equals the cost of what he takes from each of the litras, to the cost of the litra. However, the ratio of what he takes from an individual litra, to the litra, equals the ratio of what he takes from each of the litras, to the litra. Hence, the ratio of the cost of what he takes from an individual litra, to the cost of the litra, equals the ratio of the cost of what he takes from each of the litras, to the cost of the litra. By adding these up, we get that the ratio of what he takes from them all together, to the number of litras, equals the ratio of the cost of what he takes from them all together, to the cost of the number of litras. And use this as a model.

We can solve the following problem similarly. If a money-changer wants to buy variously priced coins with gold dinars, and wants to buy the same amount of each coin, how many should he buy from each?

For example, the cost of the first coin is 3 dinars per gold dinar, the cost of the second is 5 dinars per gold dinar, and the cost of the third is 7 dinars per gold dinar. The money-changer wants to buy an equal measure of each coin, with a single gold coin. With a third of the gold, he can buy a dinar's worth of the first coin; with a fifth of the gold, he can buy a dinar's worth of the other coin; and with a seventh of the gold he can buy a dinar's worth of the other coin. Hence, with a third of the gold plus a fifth and a seventh of it, he can buy a dinar's worth of each of the coins. And the ratio of the gold to these fractions, equals the ratio of what he takes from each coin, to the dinar. In this way, he should take a dinar and  $34$  of  $71$  parts of a dinar, from each coin. And use this accordingly as a model for similar problems.

This last example is correct but a little confusing, because the cost for coins is given in dinars per gold dinar. The gold dinar acts here as the standard measure for coins, just as a litra does for weight. That is, a gold dinar's worth of the first coin costs 3 dinars. The money-changer has one gold dinar, so he can get 3 dinars worth of the first coin for his gold dinar, or a dinar's worth for one third of his gold dinar. The final answer comes from inverting  $71/105$ , the sum of  $1/3$ ,  $1/5$  and  $1/7$ .

**9. Problem:** A merchant sells items of different costs. What is the smallest number of measures one can buy from each one, so that the cost of what he buys from this one equals the cost of what he buys from that one?

That is, given integers  $a_1, a_2, \dots, a_n$ , find the smallest  $b_i$ 's such  $a_1 b_1 = a_2 b_2 = \dots = a_n b_n$ .

The method is to write all the costs in a row one after the other. Beneath them, always in a single row, write one number under the other, and do so for each of the numbers. After this, take the smallest numbers corresponding to these ratios, and the result under each item's cost is the number of measures that he should buy from that item.

Levi's description is also ambiguous in the Hebrew, as attested to by the half dozen or so different figures in the various mss. The ambiguity is made worse by the words "one" (אחד) and "other"

(אח) which differ by a single letter that look alike, especially in certain hands. The translation and figure that appear represent a composite which I believe is the correct interpretation. His example that follows justifies my choice. Note how the pair for each ratio is in a new line by itself, just as Levi insisted “always in a single row”.

For example, a merchant sells four drugs. The cost of the first drug is 2 dinars per litra; the cost of the other is 3 dinars per litra; the cost of the other is 12 dinars per litra; the cost of the other is 20 dinars per litra. We want to know: how many litras should one buy from each one of the drugs, so that the cost of what is bought from one equals the cost of what is bought from the other ones?

<u>2</u>	<u>3</u>	<u>12</u>	<u>20</u>
3	2		
	12	3	
		20	12
<u>30</u>	<u>20</u>	<u>5</u>	<u>3</u>

In one line, write down the costs per litra of the items, which are 2, 3, 12 and 20; and always in a new line underneath, write the matter in the order we discussed before. That is, write 3 under 2, and 2 under 3. Similarly, write 12 under 3, and 3 under 12. Also, write 20 under 12, and 12 under 20. Afterwards, take the smallest numbers that relate according to these ratios, that is, the ratio of 3 to 2, the ratio of 12 to 3, and the ratio of 20 to 12. These numbers, according to the explanation of Euclid, are 30, 20, 5 and 3; meaning that the ratio of 30 to 20 equals the ratio of 3 to 2, and the ratio of 20 to 5 equals the ratio of 12 to 3, and the ratio of 5 to 3 equals the ratio of 20 to 12. And so we buy 30 litras from the drug that costs 2 dinars; and this is the number associated with that cost. We buy 20 litras from the drug that costs 3 dinars. We buy 5 litras from the drug that costs 12 dinars. We buy 3 litras from the drug that costs 20 dinars. The costs of what we buy from each one are all equal. And use this as a model.

Levi is asking for the smallest numbers  $x, y, z$  and  $w$  such that  $x/y = 3/2, y/z = 12/3$  and  $z/w = 20/12$ . That is,  $2x=3y=12z=20w$ . The least common multiple of 2, 3, 12 and 20 is 60; dividing by 2, 3, 12 and 20 respectively, gives  $x=30, y=20, z=5$  and  $w=3$ . At the beginning of his book, Levi warns that Books 7-9 of Euclid are prerequisite reading, and that he will not review what he expects the reader to know. In the last paragraph, Levi keeps his promise, saving himself a long discussion of least common multiples. He is implicitly referring to Book 7 Propositions 33-39.

It is appropriate to stick to this method if the cost of one of the drugs is in peshuts, since you convert the costs of the other drugs to peshuts. Use this as a model for similar things.

This is right, because the ratio of 30 to 20 equals the ratio of 3 to 2. Hence, the product of 30 with 2 equals the product of 20 with 3. However, the product of 30 with 2 equals the cost of 30 litras at 2 dinars per litra; and the product of 20 with 3 equals the cost of 20 litras at 3 dinars per litra. Hence, the cost of 30 litras at 2 dinars per litra equals the cost

of 20 litras at 3 dinars per litra. Similarly, the cost of 20 litras at 3 dinars per litra equals the cost of 5 litras at 12 dinars per litra, and the cost of 5 litras at 12 dinars per litra equals the cost of 3 litras at 20 dinars per litra. And use this as a model.

**10. Problem:** A merchant sells items of different costs, and a buyer comes to purchase a single measure from all of them, so that the cost of what he buys from one item equals the cost of what he buys from each of the other items. How much of the measure should he buy from each item? What is the cost of the measure?

The problem is the general case of: Given  $a, b, c$ , find  $x, y, z$  such that,  $x+y+z = 1$  and  $ax=by=cz$ .

It is appropriate to find the smallest number of measures to take from each item, so that the costs of each are equal. When you complete this task, add the numbers together and note the sum. This sum represents the number of parts in which to divide the measure. From each item, take the number of these parts equal to the number beneath that item, and this is what was requested.

In the last sentence, Levi is referring implicitly to Problem 9, so that when he says “beneath”, he means the numbers at the bottom of the previous figure.

However, you learn the cost of the measure, by adding up all the costs of these smallest amounts each according to its item, which is cost of their combination, and dividing by the noted sum. This is what was requested.

For example, a merchant is selling 3 drugs. The cost of the first is 7 peshuts per litra; the cost of the second is 10 peshuts per litra; the cost of the third is 20 peshuts per litra; and the buyer comes to purchase a litra from all of them, where the cost of what he buys from one equals the cost of what he buys from each of the others. The smallest amounts to take from each one so that the costs are equal, are 20, 14 and 7. Their sum is 41, and this is the number of parts in which to divide the litra. Buy 20 parts from the drug that costs 7 peshuts per litra; buy 14 parts from the drug that costs 10 peshuts per litra; and buy 7 parts from the drug that costs 20 peshuts per litra. All together, this is 41 parts, which is one litra.

This is one of many examples, that can be used effectively to teach problem solving, by taking the focus off the mechanical algebraic manipulation. As he does throughout, Levi solves this problem without “substituting” or “solving” as one might do with access to modern algebraic tools.

However, the cost of the litra is found by taking the costs of the amounts 20, 14 and 7, respectively, and adding them up. The total is 35 dinars. Divide 35 dinars by 41, to get 10 peshuts and 10 of 41 parts of a peshut, which is what was requested. And use this as a model.

This is right, because the cost of 20 litras at the low price, equals the cost of 14 litras at the intermediate price, which equals the cost of 7 litras at the high price. The ratio of 20 of 41 parts of a litra, to 20 litras, equals the ratio of 14 of 41 parts of a litra, to 14 litras, which equals the ratio of 7 of 41 parts of a litra, to 7 litras. And accordingly, the ratio of the cost of 20 of 41 parts of a litra, to the cost of 20 litras, equals the ratio of the cost of 14 of 41 parts of

a litra, to the cost of 14 litras, which equals the ratio of the cost of 7 of 41 parts of a litra, to the cost of 7 litras. By exchanging them, the ratio of 20 of 41 parts of a litra at the first price, to 14 of 41 parts of a litra at the second price, equals the ratio of the cost of 20 litras at the first price, to the cost of 14 litras at the second price. However, the cost of 20 litras at the first price, equals the cost of 14 litras at the second price. Hence, the cost of 20 of 41 parts of a litra at the first price, equals the cost of 14 of 41 parts of a litra at the second price. Similarly, the cost of 14 of 41 parts of a litra at the second price, equals the cost of 7 of 41 parts of a litra at the third price. And use this as a model.

**11. Problem:** A merchant sells two items at different prices, and a buyer wants to purchase a single measure of the two, for a given cost which is greater than the lower price and smaller than the higher price.

The question here is implicitly: What fraction of the measure should he buy from each item? The problem is equivalent to: Given  $a < b < c$ ,  $x + y = 1$  and  $ax + cy = b$ , find  $x$  and  $y$ .

It is appropriate to take the excess of the higher price over the lower price, and divide the measure into this many parts. Then take the deficiency of the lower price, to find the number of parts to buy from the higher priced item; and the excess of the higher price, to find the number of parts to buy from the lower priced item.

“Deficiency” and “excess” are with respect to the given cost.  
His solution is: let  $x = (c-b)/(c-a)$  and  $y = (b-a)/(c-a)$ .

For example, the merchant sells two drugs. The price of the first is 17 peshuts per litra, and the price of the second is 24 peshuts per litra. A buyer wants to buy a measure of the two at a cost of 19 peshuts. The excess of the higher price over the lower price is 7, so divide the measure into 7 parts. The difference between the lower price and 19 is two, so this is the number of parts of the measure that should be bought from the higher priced item. The excess of the higher price is five, so this is the number of parts that should be bought from the lower priced item. The cost of the measure is 19 peshuts. And use this as a model.

This is right, because here we have three different numbers: 17, 19 and 24. The product of 24 with two, which is the excess of the middle over the small, plus the product of 17 with five, which is the excess of the large over the middle, counts 19, the middle, as many times as the excess of the large over the small, which is 7.

Levi is implicitly referring to Theorem 45 in part one of the book which states that given  $a < b < c$ , then  $c(b-a) + a(c-b) = b(c-a)$ . See Appendix.

Hence, the sum of the cost of two litras at the price of 24 peshuts each, plus the cost of five litras at the price of 17 peshuts each, equals 7 times 19. The number of litras here is 7, hence each one sells for 19 peshuts. Furthermore, the ratio of two sevenths of a litra, to two litras, equals the ratio of five sevenths of a litra, to five litras. Accordingly, the ratio of the cost of two sevenths of a litra at the higher price, to the cost of two litras at the higher price, equals the ratio of the cost of five sevenths of a litra at the lower price, to the cost of five litras at the lower price. Accordingly, the ratio of the cost of two sevenths of a litra at the

higher price plus five sevenths of a litra at the lower price, to the cost of two litras at the higher price plus five litras at the lower price, equals the ratio of the cost of two sevenths of a litra at the higher price, to the cost of two litras at the higher price.

Levi is using the well known theorem that if  $a/b = c/d$ , then  $(a+c)/(b+d)$  is also equal to them. He discusses this theorem and similar ones in Section f of part two.

However, the ratio of the cost of two sevenths of a litra at the higher price, to the cost of two litras at the higher price, is a seventh of the cost. Hence, the ratio of the cost of two sevenths of a litra at the higher price plus five sevenths of a litra at the lower price, to the cost of two litras at the higher price plus five litras at the lower price, is a seventh of the cost. It was already made clear that the total cost is 7 times 19. Hence, the cost of this litra is a seventh of 7 times 19, which is 19 peshuts. And use this as a model.

When Levi says that the ratio is “a seventh of the cost”, he means that the former cost is  $1/7$  the latter cost. That is, the ratio itself is equal to  $1/7$ . Also, notice that Levi uses “litra” here instead of the more general “measure” used in the statement of the problem.

**12. Problem:** A merchant sells a number of items with different prices. A buyer comes to purchase a single measure from them all, whose total cost exceeds the lowest price and is exceeded by the higher price. What parts of the measure should he buy from each item?

This is a generalization of the previous problem from two variables to  $n$ . Levi knows that his solution is not unique, and he implies this by his comment in the next paragraph “so that the answer will be as small as possible”. He also implies that any solution must include a positive measure of each item. That is, no item can be left out.

It is appropriate to pair up each one of the lower prices with one of the higher prices. If the higher prices do not suffice with respect to the lower, since the number of higher prices is either greater or smaller, then match up each of the corresponding remaining prices, with the item whose price is closest to the desired cost, so that the answer will be as small as possible. After you complete this procedure as described for all the pairs, then sum up all the excesses of the larger number over the smaller number in each pair, and divide the litra into this many parts. If there was an item whose price is equal to the desired cost, so that this item has no pair, then take one part or many parts from this item, accordingly as you wish. Add this number of parts to the sum, and divide the litra into that many parts.

For example, the merchant sells seven drugs. The price of the first drug is 3 peshuts per litra; the price of the second drug is 5 peshuts; the price of the third drug is 8 peshuts; the price of the fourth is 11 peshuts; the price of the fifth is 15 peshuts; the price of the sixth is 19 peshuts; and the price of the seventh is 28 peshuts. A buyer wants to purchase a litra from all with 15 peshuts. There are two higher prices: the drug that costs 19 peshuts per litra, and the drug that costs 28 peshuts per litra; and there are four lower prices: the drug that costs 4 peshuts, the one that costs 5 peshuts, the one that costs 8 peshuts, and the one that costs 11 peshuts. We place 3 peshuts corresponding to 19 peshuts, and 5 peshuts corresponding to 28 peshuts. Since there are leftover drugs with lower prices but none with

higher prices, we take the higher price that is closest to the desired cost 15, and this is 19. We pair up 19 with each of the remaining lower prices, as you can see in this figure.

$$\begin{array}{cccc} \underline{3} & \underline{5} & \underline{8} & \underline{11} \\ \underline{19} & \underline{28} & \underline{19} & \underline{19} \end{array}$$

We already know how to take a litra from each of these pairs, whose cost is 15 peshuts. Accordingly, take 4 of 16 parts of a litra from drug 3, and 12 of 16 parts of a litra from drug 19. Also, take 13 of 23 parts of a litra from drug 5, and 10 of 23 parts of a litra from drug 28. Furthermore, take 4 of 11 parts of a litra from drug 8, and 7 of 11 parts of a litra from drug 19. Also, take 4 of 8 parts of a litra from drug 11, and 4 parts of 8 from drug 19. The number of all these parts is 58, so we add two parts for drug 15, that has no pair. We divide the litra up into the resulting number of parts which is 60. We take 4 parts from drug 3, 13 parts from drug 5, 4 parts from drug 8, 4 parts from drug 11, two parts from drug 15, 23 parts from drug 19, and 10 parts from drug 28. The cost of the litra is 15 peshuts, and this is what was requested.

This is right, because it is clear from what preceded, that the cost of 4 litras at price 3, plus 12 litras at price 19, equals 16 times 15. The matter is similar for each pair of corresponding numbers. Furthermore, it is obvious that the cost of 2 litras at price 15 is equal to 2 times 15. By adding everything together, the cost of all these litras, each of which is a multiple of 15, equals 60 times 15.

However, the ratio of one of 60 parts of a litra, to a litra, equals the ratio of the price of one of 60 parts of a litra, to the price of a litra. It is clear by our previous explanation, that the ratio of the price of the sum of all the parts, to the price of the sum of the litras, which is 60 times 15, equals the ratio of one to 60. Thus, the price of the parts all together is 15 peshuts, which is one of 60 parts, of 60 times 15. And use this as a model.

**13. Problem:** One man hires another to work a given number of days, for a fixed wage. This job requires the hiring of a certain number of men per day, each of whom leads a certain number of animals, each of which carries a given number of measures and walks a given distance. The hired man deviates from some or all of these numbers. How much should his wages be?

Take the product of all the values that were stipulated, and make a note of it. Furthermore, take the product of the actual values that were accomplished; and the ratio of the number you noted, to this product, equals the ratio of the wages he promised him, to the wages he owes him.

For example, Reuven hired Shimon to work 9 days for 10 litra. The job stipulated the hiring of 13 men each day, each of whom leads seven animals, each of which carries 15 measures and walks 6 parsas. Shimon provided 8 days, 17 men, each of whom led 6 animals, each of which carried 11 measures and walked 7 parsas. The product of the stipulated numbers 9, 13, 7, 15 and 6, equals 73 thousand and 710, which is noted. The product of the accomplished numbers, 8, 17, 6, 11 and 7, equals 62 thousand and 832. The ratio of 10 litra

to what he owes him, equals the ratio of the noted value to 62 thousand and 832. If you multiply 10 litra, the first number, by the fourth number, which is 62 thousand and 832, and you divide the result by the noted value, you will get the number of litras and fractions of a litra that he owes him. This is 8 and one half litra; and a thousand and 7 hundred and 85, of 62 thousand and 832 parts of a litra, which is 5 peshuts and 59 thousand and 850, of 73 thousand and 710 parts of a peshut.

Recall that there are 20 dinars in a litra, and 12 peshuts in a dinar.

Note the error here. The last 62,832 in the previous paragraph should be 73,710. This careless error is found in all the extant mss. There is no way to know for sure whether it is due to the scribes or to Levi himself.

This is right, because the ratio of what he owes to what he stipulated, equals the ratio of what he did, to what he agreed to do. And the ratio of what he did, to what he agreed to do, is composed of the ratios of the numbers that were stipulated, to the corresponding numbers that were accomplished. This composed ratio, as we already explained, equals the ratio of the product of the stipulated numbers, to the product of the numbers that were accomplished. And use this as a model.

Accordingly, this explains: if a man stipulated to another man, to fill a container of given length, width and depth with a particular item, for a given amount of money; and he filled a different container with various different dimensions from what he stipulated; how much does he owe him? Here, the ratio of the product of the three stipulated dimensions, to the product of the three actual dimensions, equals the ratio of the stipulated cost, to the amount he owes him. And judge accordingly for similar cases.

For example, the seller sold the buyer a full container of oil for 20 dinars. The length of the container was ten measures, its width was nine and its depth was 12. He actually filled a container whose length was eleven, width was six and depth was 17. The product of the former dimensions is a thousand and 80, and this is noted. The product of the actual dimensions is a thousand and 122. The ratio of the noted value to a thousand and 122, equals the ratio of 20 dinars to what he owes him.

This last example is one of many where we presumably get information about the costs of goods and services in 13<sup>th</sup>-14<sup>th</sup> century Provence.

For units of length, Levi uses the generic word for measure, rather than any specific unit. This is unlike weight and money where he uses the specific units of litra, dinar and peshut. Perhaps, units of length were not as standardized in his day as was weight or money.

**(Ed. 1) 14. Problem:** Reuven bought an item at a given rate of certain fractions of a measure for a certain number of dinars, and sold at another rate of certain fractions of the measure for a certain number of dinars. The capital is given. We want to know if he earned or lost, and how much?

By “capital”, Levi means the amount he spent.

The method is to find the price of a measure in dinars, at each rate. It will be then be clear to you whether he lost or earned with each measure, and how many dinars per measure. The total number of measures will thereby also be clear. Multiply the total number of measures, by what he earned or lost per measure, and the result is the total number of dinars that he earned or lost.

If the problem were reversed, that is, he earned or lost a certain number of dinars, and you want to know the capital or the number of measures purchased, then divide the number of dinars that he earned or lost from the whole transaction, by what he earned or lost per measure. The result is the total number of measures purchased, and their value at the buying rate, is the capital.

A special case of this “reverse” problem shows up as problem 14 in the second edition.

For example, Reuven bought 2 fifths and 3 sevenths of a measure for 7 dinars and 8 of 11 parts of a dinar. He sold 4 ninths of a measure for 8 dinars and 3 sevenths of a dinar. The capital was a hundred dinars. We want to know whether he lost or earned, and how much?

The value of a measure at the buying rate is 9 dinars and 104 of 319 parts of a dinar. The value of a measure at the selling rate is 18 dinars and 27 of 28 parts of a dinar. Accordingly, the total number of measures purchased is 10 measures, and 21 hundred and 50, of 29 hundred and 75 parts of a measure, which is 86 of 119 parts of a measure. So he earned 9 dinars and 5 thousand 701, of 8 thousand 932 parts of a dinar, per measure. Multiply this by the number of measures, to find that he earns 103 dinars and 370 thousand and 40, of a thousand and 62 thousand 908 parts of a dinar, which is 3 thousand 190, of 9 thousand 163 parts of a dinar.

Note that “a thousand and 62 thousand 908” means 1,062,098. To appreciate this, pause before and after reading the second “thousand”.

If the problem was that he earned or lost 100 dinars in this transaction, and you wanted to know the capital, then divide 100 dinars by what he earns per measure, which in our example is 9 dinars and 5 thousand 701, of 8 thousand 932 parts of a dinar. The result is the number of measures, which is 10 measures and 32 thousand and 310, of 86 thousand and 89 parts of a measure. If you multiply the number of measures, by the total price per measure at the buying rate in dinars, the result is the capital. The reason for this is completely explained by our previous discussion.

**(Ed. 2) 14. Problem:** A man bought an item at a given rate of a certain fraction or fractions of a measure or litra, for a certain number of dinars. He sold at another rate, of certain fractions of a measure, smaller than the first, for the exact same number of dinars, and thereby made a profit. What is the capital?

This problem is a special case of the “reverse” problem from Problem 14 of the first edition. The method used here is different from the method used in the first edition. The first edition works with an



idea based on the calculations of dinars per measure, while the second edition uses measures per dinar. However, both editions describe similar problems and discuss three variations of their respective original problems.

For example, Reuven bought at the rate of 2 fifths and 3 sevenths of a measure, for 2 dinars. He sold at the rate of 4 ninths for 2 dinars, and made 100 dinars profit. We want to know: what is the capital?

The method is to take the common denominator of these fractions both from the measure and from the dinars, if indeed there are fractions of either. The common denominator of these fractions in our example is 315, and that is the measure. Take 2 fifths and 3 sevenths of it, to get 216.

This is an error. The result 216 is wrong; it should be 261. Note that changing the buying rate to  $2/5 + 2/7$  makes 216 correct. This is more likely the source of the error, than a careless digit swap, since in the Hebrew numbering, 216 (ריו) and 261 (אסו) do not look alike. The rest of the calculation is consistent with this error, and thereby incorrect. Unlike previous errors, this one propagates forward with consistency, implying that it was likely in the original, and not the fault of the scribes.

Hence, he buys 108 of these parts for a dinar. Also, take 4 ninths of it to get 140. Hence, he sells 70 of these parts for a dinar. Accordingly, he profits 38 of these parts with each dinar. We already know that he earned 100 dinars, which is 7 thousand of these parts.

Since he sells at the rate of 70 parts per dinar, 100 dinars is “worth” 7000 parts.

It is appropriate that we take a number whose ratio to 108, equals the ratio of 7 thousand to 38, and it is already clear how to do this. If you divide this result by 70, you find the capital plus the profit together. If you divide the result by 108, which is the total number of parts that he bought for a dinar, then you get the capital. The reason for this is clear from the preceding discussion.

“This result” is the number of parts that he bought. The number of parts he bought, divided by the number of parts he can sell for a dinar, equals the value he can get for the parts bought, equals the money he received, equals the total he spent plus total profit. The number of parts he bought, divided by the number of parts he can buy for a dinar, equals the value he spent on the parts bought, which is the capital.

Notice that this method is different from the one used in the “reverse problem” of Problem 14 in the first edition. The capital in the first edition equals (total measures purchased) \* (price in dinars per measure); while in the second edition, the capital equals (total parts of measure purchased)/(parts of a measure he gets for a dinar). Perhaps this change is made in order to avoid the big fractions that appear due to the method of the first edition, or perhaps because this method generalizes more easily as seen immediately below.

If we posed the opposite problem: that is, he lost 100 dinars by buying 4 ninths of a measure at 2 dinars, and selling 2 fifths plus 3 sevenths at 2 dinars; then the method is the same, except that when you divide the result by 70, you get the capital, and if you divide it by 108, you get what remains after the loss.

Because the buying rate is now 70 parts per dinar, and the selling rate is 108 parts per dinar.

If you ask for the quantity he buys, then divide the result by 385, which is the number of parts in the measure, and you get the quantity he bought in measures. And use this as a model.

This is an error. The number 385 should be 315. The numbers do not look alike (שטר and שפה), but perhaps the 385 comes from Problem 15 in the second edition, where  $385 = 5 \cdot 7 \cdot 11$  is used in another profit/loss example.

If the number of dinars were not equal: that is, he buys a certain fraction of a measure at 3 dinars, and sells some fraction of it at 25 dinars, and profits or loses 100 dinars. Using the previous method, find the parts of the common denominator that you can buy for a dinar, and the parts that you can sell for a dinar. Continue in the manner of the previous method, and you will get what was requested.

This generalization is the same as the “reverse problem” of Problem 14 in the first edition.

If the parts were one and the same, but the number of dinars varied: for example, he buys 4 of 7 parts of a litra, plus 5 of 11 parts of it, for 7 dinars and 8 of 11 parts of a dinar; and he sells these parts for 9 dinars and 3 of 13 parts of a dinar; and he profits 100 dinars; and the problem posed is: what is the capital, or what is the quantity he bought? The method is to take the common denominator of these parts of the litra, and divide the litra into this many parts. You will derive the total number of parts he buys with a dinar, and the total number of parts he gives for a dinar. Then continue the matter in the previous fashion, to know the capital and to know the quantity he bought.

The word (בה) “of it” in the phrase “5 of 11 parts of it”, is written (בה) which means “with 5” in all three of the second edition mss., however, the slash on the leftmost letter is an error.

And if someone asks: What is the amount, based on a fixed measure, such that 2 sevenths plus 3 fifths of it exceeds, or is exceeded by, 4 ninths of it, by the fixed measure? The method is explained in Theorem 49.

This reference reflects the second edition numbering of the theorems in part one. Theorem 49 in the second edition does indeed solve this problem, while Theorem 49 in the first edition is unrelated. Please see the list of the theorems from part one in the Appendix.

**(Ed. 1) 15. Problem:** Reuven bought certain fractions of a measure, for a certain number of dinars, and the capital was a certain number of dinars. He sold part of the purchase at a different rate of certain fractions of a measure for a certain number of dinars; and he sold the rest at another rate of certain fractions of a measure for a certain number of dinars. He neither earned nor lost. We want to know: what part of the purchase did he sell at the first rate, and what part did he sell at the second rate?

This problem reduces to finding  $x$  and  $y$ , given  $a$ ,  $b$  and  $c$ , such that  $x+y = c$ , and  $ax = by$ . In this formalization,  $a$  and  $b$  are the two selling rates, and  $c$  is the quantity bought which equals the capital divided by the buying rate.

The solution is given in the paragraph below as  $x = cb/(a+b)$ , and  $y = c - cb/(a+b)$ .

If the problem is correct, then it must be the case that he earns with the first rate and loses with the second. Find the cost of a measure at each rate, and see how much he loses per measure with the first, and how much he gains per measure with the second. The ratio of the loss to the gain, equals the ratio of what he sold at the higher rate to what he sold at the lower rate. This will be clear with a slight investigation of Theorem 39 in part one.

Theorem 39 in the first edition is completely unrelated. Neither Theorem 39 in the second edition nor any other theorem seems at all related. It is not clear what Levi intended by this reference.

By adding these up, the ratio of the loss, to the sum of the gain and the loss, equals the ratio of what he sold at the higher rate, to the size of the whole purchase. Since you know three of these, you can find the fourth in the fashion presented at the start of this section.

For example, Reuven bought at the rate of 2 sevenths of a measure for 3 dinars and a fourth of a dinar, and the capital was 100 dinars. He sold part of the purchase at the rate of 4 ninths of a measure for 6 and a third dinars. He sold the other part at the rate of 5 of 11 parts of a measure for 4 dinars and a fifth. The cost of a measure at the buying rate is 11 dinars and 3 eighths of a dinar. The value of a measure at the first rate is 14 dinars and a fourth of a dinar. At the second rate, the value is 9 dinars and 6 of 25 parts of a dinar. Accordingly, the quantity purchased is 8 measures and 72 of 91 parts of a measure.

This last value equals the capital, 100, divided by the buying rate,  $11 \frac{3}{8}$ . The last word, "measure", is clearly the correct word. However all first edition mss. have a careless error and use "dinar" instead. The careless error could be either original or due to scribes.

The earnings per measure, according to the first rate, is 2 dinars and 7 eighths of a dinar. The loss per measure, according to the second rate, is 2 dinars and 27 of 200 parts of a dinar. The sum of the earnings and the loss is 5 dinars and 2 of 200 parts of a dinar. Multiply the loss per measure, in dinars and fractions thereof, by the total amount purchased, in measures and fractions thereof; and divide by the sum of the earnings and the loss, in dinars and fractions thereof. The result is the total amount purchased, in measures and fractions thereof, that were sold at the higher rate. Multiply 2 whole and 27 of 200 parts of one, by 8 whole and 72 of 91 parts of one. Divide the result by 5 whole and 2 of 200 parts of one, to get the total amount purchased, in measures and fractions thereof, that were sold at the higher rate. The remainder of the measures purchased were sold at the lower rate.

Many variations are now discussed.

If the problem was that he earned or lost some number of dinars, then extract the total number of measures which he earned or lost, from the total number purchased, and make a note of it. Split what remains of the purchase in the previous fashion, and find the amount sold at the higher rate. The leftover is what he sold at the lower rate. If he lost, then add the amount sold at the lower rate, to the noted value. If he earned, then add the noted value to the amount sold at the higher rate.

If the problem was that he sold 20 measures at the higher rate, and we want to know how many measures he should sell at the lower rate in order that he neither earns nor loses, then the ratio of 20 measures to the unknown, equals the ratio of 2 dinars and 27 of 200 parts of a dinar, to 20 dinars and 7 eighths of a dinar. Accordingly, you can find the unknown number in the previous fashion.

The preceding discussion explains the following. If a man trades one item for another; and the value of certain fractions of the item he offers is a certain number of dinars; and the value of different fractions of the item he receives is another number of dinars; what does he deserve to receive? For each item, find the value of one measure. The ratio of the value of a measure for the item he receives, to the value of a measure for the item he offers, equals the ratio of the total measures he receives, to the total measures he offers. Three of these numbers are known, so you can find the fourth.

Also, if you know the sum total of the measures offered and received, you can extract how many measures he offered and how many he received. Take the price per measure for each item. The ratio of the price per measure of the item he offers, to the sum of the prices per measure of what he offers and receives, equals the ratio of the measures he receives to the sum total of all the measures he offers and receives. Three of these numbers are known, so you can find the fourth.

Also consider, if a man traded one item for two items; the price of certain fractions of a measure of the item he offers, is a certain number of dinars; the price of other fractions of a measure of one of the items he receives is given; the price of other fractions of a measure of the two items he receives is given; he offers a certain number of measures; he receives a certain number of measures of the two items; and we want to know: how many measures did he receive from the first item, and how many from the second?

Take the total value of what he offered and divide it by the number of measures he receives. This will be the combined value per measure of these two items. You already know the method for deciding what part of a measure to take from the first item, and what part from the second item, such that the value of the measure will be a fixed amount.

The method referred to in the last paragraph is from Problem 11.

Accordingly, multiply the part of the measure taken from the item one, by the number of measures taken from both items. The resulting fractions are the fractions of a measure he receives from this item, and the remainder is what he receives from the other item. The reason for this is clear from the previous discussion.

For example, Reuven gives Shimon, 7 measures of a certain item, that costs 2 dinars and a seventh per measure; he receives 9 measures of two different items from Shimon, the cost of the first is a dinar and a half per measure, and the cost of the second is 2 dinars and 3 quarters per measure. The value of the 9 measures he received is equal to the value of the 7

measures he gave. We want to know: how many measures did he receive from the first item and how many did he receive from the second?

We know that the value of what he received is 15 dinars.

This is because the seven measures he gave are worth  $2\frac{1}{7}$  dinars per measure, and we are told that these seven measures total are worth the same as the nine measures he received.

Accordingly, the value per measure of what he receives, is a dinar and  $\frac{2}{3}$  thirds. Since the value per measure of the first item is a dinar and a half, and that of the second item is 2 dinars and  $\frac{3}{4}$  fourths, he should take 2 of 15 parts of a measure from the item whose value is 2 dinars and  $\frac{3}{4}$  fourths, and 13 parts of a measure from the other item. And the value of a measure is a dinar and  $\frac{2}{3}$  thirds of a dinar. This is made clear by converting all the fractions of a dinar to parts of the same kind, that is, the third, the half and the fourth. Thus the deficit of the smaller is 2 of these parts and the excess of the greater is 13 of these parts. Consequently, it is clear that he should take 7 measures and  $\frac{4}{5}$  fifths of a measure from the item whose value is a dinar and a half per measure, and take the rest from the other item.

The variations that follow are generalizations of Problem 13.

Similarly, if a man stipulated to pay another man a certain wage in dinars for a certain amount of time; and for each day that he misses, the worker pays a certain number of dinars to the hirer; the worker works a certain number of days and misses a certain number; and his loss exceeds his gain; how much time did he miss?

Find the total number of dinars that the worker earns in a working day, and the total number of dinars that he loses in a missed day. The ratio of the total dinars earned to the total dinars lost, equals the ratio of the time he missed to the time he worked. This is self evident. Now that three of these numbers are known, you can extract the fourth from them. Also, if the sum of the two times was known, you could find out how much of the time he worked and how much he missed, in the previous fashion.

**(Ed. 2) 15. Problem:** A man traded an item for another. The price of the item he gave is fixed for certain fractions of a measure. The price of the item he received is fixed for certain fractions of a measure,. He gave a certain measure. How much should he receive?

For example, the item he gave costs 2 dinars for  $\frac{3}{5}$  fifths plus  $\frac{2}{9}$  ninths of a measure. The item he received costs 3 dinars for  $\frac{3}{11}$  parts of a measure plus  $\frac{2}{7}$  sevenths of a measure. He gave him 20 measures. We want to know: what should he receive?

The method is to take the common denominator of these parts, both from the measure and from the dinar, to the extent that there were any such parts. For each price, find the number of parts that are worth a dinar. The ratio of these parts for the item given, to these parts for the item he received, equals the ratio of the quantity given to the quantity received. The common denominator of all these parts is 3 thousand and 465. Extract  $\frac{3}{5}$  fifths and  $\frac{2}{9}$  ninths to get 2 thousand 464.

This is an error. The number 2464 should be 2849. It would be correct if the fraction  $\frac{2}{9}$  was changed to  $\frac{3}{9}$ . The rest of the problem is consistent with this error and thereby incorrect. Because the error propagates on, it is likely that it appeared in the original and unlikely to be the fault of the scribes.

Accordingly, for the item he gave, the number of parts per dinar is a thousand and 232. Extract 2 sevenths and 3 of 11 parts of a measure, to get a thousand and 935. Accordingly, for the item he received, the number of parts per dinar is 645. The ratio of a thousand and 232, to 645, equals the ratio of what he receives, to 20 measures; and you already know the method to get the missing number. The reason for this is clear. And use this as a model.

As in Problem 14, the distinction between Problem 15 of the first edition and Problem 15 of the second edition, is that the first edition uses dinars per measure, and the second edition uses measures per dinar.

Similarly, it is clear: if a man stipulates to pay another man a certain wage in dinars, for a certain time period of work, and the worker gives up after part of the time, and his loss exceeds his gain, how much was the loss?

This is a special case of Problem 13.

For example, Reuven hired Shimon to do a job lasting 7 days and a fifth of a day, for 20 dinars. Shimon gives Reuven 4 of 11 parts of a dinar for each day he misses. Shimon worked 2 days and 3 of 7 parts of a day, gave up, and his loss exceeded his gain. We want to know: how much did he lose?

We take the common denominator of all these parts to get 385, and divide the day into this many parts.

The 385 is correct here. However, in Problem 14 of the second edition, 385 appears incorrectly in place of 315. It may have been incorrectly copied from here. See commentary there.

Then find the number of these parts for which he earns a dinar, and the number of these parts for which he loses a dinar. The ratio of the number for which he earns, to the number for which he loses, equals the ratio of 2 days and 3 sevenths, to the time he missed. He earns a dinar with each 138 parts and 12 of 20 parts of a part. He loses a dinar with each thousand and 58 parts and 3 of 4 parts of a part. And such is the ratio of 2 days and 3 of 7 parts of a day, to the time he missed. And use this as a model.

Note again the consistent style in the second edition of day per dinar rather than dinar per day. Note also, that Levi was not averse to using parts of a part.

And if someone asked: What is the number, such that given fractions of it, equal different given fractions of a given number? The method is shown in Theorem 51.

This reference reflects the second edition numbering of the theorems in part one. Theorem 51 in the second edition does indeed solve this problem. Theorem 51 in the first edition is unrelated. Please see the list of the theorems from part one in the Appendix.

### **VIII. A Critical Edition of Problems 1-15**

We present a critical edition in Hebrew of Problems 1-15 using all twelve extant mss. The critical apparatus appears only in the Hebrew. The English translation and commentary repeat and expand on the important points, but do not duplicate every dull detail. On the other hand, the commentary which appears only in English, is often necessary to decipher the difficult passages. Any errors found in all extant mss. are footnoted, and commentary on these can be found in the translation.

The critical apparatus attempts to be comprehensive but for practical considerations I leave out certain types of changes that vary widely and randomly, and do not effect the meaning of the passage. These include variations of feminine and masculine forms in verbs and nouns, variations of writing numbers one through ten as a word or as a letter, and trivial variations in abbreviations, spelling or plural formations (for example, *mem* versus *nun*). Despite this, the apparatus is still quite dense and includes many variations, that may not interest the casual reader. If the scribe made correct corrections or changes, then these are not noted. However, if the correct corrections were made by a subsequent owner or editor, then the mistake or omission, and correction are footnoted. Various tables and figures are found in some of the mss. Because of the broad variation and the fact that these are not textual, I present the best composite.

Footnotes appear with comments at the bottom of each page. All punctuation, paragraphs and numbering have been added for the sake of the reader, and to allow easier cross referencing with the translation. Slashes or marks above letters are reproduced carefully, with letters that represent numbers printed normally, and letters that represent variables printed in bold, see the discussion in section V. These marks are also used at the end of a word that is abbreviated. This happens often with the words Dinar, Litra, Peshut and their plural forms.

## Appendix: List of the Theorems from Part One of *Maaseh Hoshev* in the Two Editions, with Brief Notes.

The regular numbering is from the first edition, and the numbering in parentheses is from the second.

- 1. The product of 2 numbers  $a$  and  $b$ , is  $a$  added to itself  $b$  times.  
(No proof).
- (1) 2. The product of a number  $a$  and another  $b = b_1 + b_2 + \dots + b_n$  is  $ab_1 + ab_2 + \dots + ab_n$ .  
(Proof just unravels the theorem using 1).
- (2) 3. The product of 2 numbers  $a = a_1 + a_2 + \dots + a_m$  and  $b = b_1 + b_2 + \dots + b_n$  is  
 $a_1b_1 + a_1b_2 + \dots + a_1b_n + a_2b_1 + a_2b_2 + \dots + a_2b_n + \dots + a_mb_1 + a_mb_2 + \dots + a_mb_n$ .  
(Proof just unravels the theorem using 2).
- (3) 4. The product of a number  $a = b + c$  with  $b$  is equal to  $b^2 + bc$ .  
(Corollary of 2, proof is immediate).
- (4) 5.  $(a/2 + b)$  squared is equal to  $(a+b)b + (a/2)^2$   
(Proof uses 3).
- (5) 6.  $(a+b)^2 = a^2 + b^2 + 2ab$ .  
(Proof uses 3).
- (6) 7.  $(a+b)^2 = a(a+b) + ab + b^2$   
(Proof uses 3).
- 8. If  $a = b+c$ , then either  $(a/2)^2 = bc + (b - a/2)^2$  or  $(a/2)^2 = bc + (c - a/2)^2$ .  
(Proof uses 3. Does not consider the negative case.)
- (7) 9.  $a(bc) = b(ac) = c(ab)$ .  
(Proof uses 1).
- (8) 10.  $a(bcd) = b(acd) = c(abd) = d(abc)$ .  
(Proof uses induction and 9, and explicitly implies a more general theorem, that you can take any  $n$  numbers, and their product will be the same as the product of any  $n-1$  terms times the remaining term).
- (9) 11.  $a(bcd) = (ac)(bd)$ .  
(Proof uses 10, and explicitly implies a more general theorem, that you can take any  $n$  numbers, and their product will be the same as the product of any  $n-2$  terms times the product of the remaining two terms).
- (10) 12.  $a(b_1)(b_2)\dots(b_n) = b_1(a)(b_2)(b_3) \dots (b_n) = b_2(a)(b_1)(b_3) \dots (b_n) = \dots = b_n(a)(b_1)(b_2)\dots(b_{n-1})$ .  
(Completely generalizes 10 and 11).
- (11) 13. The ratio  $((a_1)(a_2)\dots(a_n))/((b_1)(b_2)\dots(b_n)) = (a_1/b_1) (a_2/b_2) \dots (a_n/b_n)$ .
- (12) 14. The ratio  $((a_1)(a_2)\dots(a_n))/((b_1)(b_2)\dots(b_n))$  equals the product of  $a_i/b_j$ , where each  $i$  and  $j$  between 1 and  $n$  appears exactly once.
- 15. If  $a$  is relatively prime to  $b = (b_1)(b_2)\dots(b_n)$ , then  $a$  is relatively prime to  $b_i$ , for all  $i$  from 1 to  $n$ .



- 16. A number  $a$  that is relatively prime to all the integers less than  $\lceil \sqrt{a} \rceil$ , is prime.  
 ( $\lceil \sqrt{a} \rceil$ , is literally:  $\sqrt{x}$  where  $x$  is the first square larger than  $a$ .)
- 17. If one takes a fraction of a given number, and then a fraction of the remainder and continues arbitrarily, then the final remainder will be the same no matter what order the fractions were taken. Also, the sum of all the pieces taken will be the same.
- 18. If one takes a fraction of a given product, and then a fraction of the remainder and continues arbitrarily, then the final remainder will be the same no matter what order the fractions were taken.
- (13) 19. The number of terms in the sum  $1+2+\dots+n$ , is equal to the number of 1's in  $n$ .
- (14) 20. The number of odd terms in the sum  $1+2+\dots+2n$  is equal to the number of even terms.
- (15) 21. In the sum,  $n+(n+1)+(n+2)+\dots+(n+m)$ , the last term is  $m$  greater than the first.
- (16) 22. In the sum  $(n-m)+(n-m+1)+\dots+n+(n+1)+\dots+(n+m)$ , the last term exceeds the middle term by the amount that the middle term exceeds the first term.
- (17) 23. In the sum  $(n-m)+(n-m+1)+\dots+n+(n+1)+\dots+(n+m)$ , the first term is odd iff the last term is odd.
- (18) 24. If  $a-1=c-b$ , then  $a+b=c+1$ .
- (19) 25. If  $a-c=c-b$  then  $a+b=2c$ .
- (20) 26.  $1+2+\dots+n$ , where  $n$  is even, is equal to  $(n/2)(n+1)$ .  
 (Literally, half the number of terms times the number of terms plus 1. Proof works from outside in, in pairs showing that each pair sums to  $n+1$ , and there are  $n/2$  pairs).
- (21) 27.  $1+2+\dots+n$ , where  $n$  is odd, is equal to  $((n+1)/2)n$ .  
 (Literally the middle term times the number of terms. Proof works from inside out, in pairs showing that each pair sums to twice the middle term.)
- (22) 28.  $1+2+\dots+n$ , where  $n$  is odd, is equal to  $(n/2)(n+1)$ .  
 (Literally, half the last term times the number after the last term. Proof uses proportions, algebra like idea and 21).
- (23) 29.  $1+3+5+\dots+(2n-1)=n^2$ .  
 (Literally, the square of the middle term. Proves it first for an odd number of terms, then an even number).
- (24) 30.  $(1+2+\dots+n)+(1+2+\dots+n+(n+1))=(n+1)^2$ .
- (25) 31.  $2(1+2+\dots+n)=n^2+n$ .  
 (Proof uses 30).  
 (Corollary: The sum  $1+2+\dots+n=n^2/2+n/2$ .)
- (26) 32.  $1+(1+2)+(1+2+3)+\dots+(1+2+\dots+n)=2^2+4^2+6^2+\dots+n^2$ ,  $n$  even; and  $1^2+3^2+5^2+\dots+n^2$ ,  $n$  odd.  
 (Proof uses 30).
- (27) 33.  $(1+2+3+\dots+n)+(2+3+4+\dots+n)+\dots+n=1^2+2^2+3^2+\dots+n^2$   
 (Proof uses a counting argument).

- (28) 34.  $(1+2+3+\dots+n)+(2+3+4+\dots+n)+\dots+n+1+(1+2)+(1+2+3)+\dots+(1+2+\dots+(n-1))=n(1+2+3+\dots+n)$   
(Proof uses a counting argument).
- (29) 35.  $(n+1)^2+n^2-(n+1+n)=2n^2$
- (30) 36.  $(1+2+3+\dots+n)+(2+3+4+\dots+n)+\dots+n-(1+2+3+\dots+n)=2(2^2+4^2+6^2+\dots+(n-1)^2)$ ,  $n-1$  even; and  $2(1^2+3^2+5^2+\dots+(n-1)^2)$ ,  $n-1$  odd.  
(Proof uses 33 and 35).
- (31) 37.  $n(1+2+3+\dots+(n+1))=3(1^2+3^2+5^2+\dots+n^2)$ ,  $n$  odd; and  $3(2^2+4^2+6^2+\dots+n^2)$ ,  $n$  even.  
(Proof uses 32, 34 and 36).
- (32) 38.  $(n-(1/3)(n-1))(1+2+3+\dots+n)=1^2+2^2+3^2+\dots+n^2$   
(Proof uses 32, 33, 34 and 37).
- (33) 39.  $(n^2-n)/2=(1+2+\dots+(n-1))$   
(Proof uses 30).
- (34) 40.  $(n^2-n)/2+n=(1+2+\dots+n)$   
(Proof uses 30).
- (35) 41.  $(1+2+3+\dots+n)^2=n^3+(1+2+3+\dots+(n-1))^2$   
(Proof uses 30 and 6).
- (36) 42.  $(1+2+3+\dots+n)^2=1^3+2^3+3^3+\dots+n^3$   
(Proof by induction using 41).
- (37) 43. Let  $m=1+2+3+\dots+n$ , then  $1^3+2^3+3^3+\dots+n^3=1+3+5+\dots+(2m-1)$ .
- (38) 44.  $ab+a=(b+1)a$ , and  $ab+b=(a+1)b$ .
- (39) 45. Given  $a<b<c$ , then  $c(b-a)+a(c-b)=b(c-a)$ .
- (40) 46. Given  $2<a<b$ , then  $2(a-2)(b-1)+b+(a-2)+(b-a)=2(a-1)(b-1)$ .  
(Minor differences in the two editions).
- (41) 47. Given  $a<b$ , then  $ba+(b-a)=(a-1)(b-1)+b+(b-1)$ .
- (42) 48. Given  $2<a<b$ , then  $2(b-1)(a-2)+b+(a-2)+(b-a)=(a-1)b+(a-2)(b-1)+(b-a)$ .  
(Minor differences in the two editions).
- (43) 49. Given  $a<b<c$ , and  $d=a-2$ , then  $2(c-1)(b-a)+cd+(c-1)d+c+(b-a)+(c-b)=2(b-1)(c-1)$ .  
(Minor differences in the two editions).  
(Generalizes 46).
- (44) 50. Given  $a<b<c$ , and  $d=a-2$ , then  $2(c-1)(b-a)+(cb)d+(c-b)(a-1)+c+(b-a)=(b-1)(c-1)a$ .  
(Major differences in the two editions).
- (45) 51. Given  $a<b<c$ , then  $(c-1)(b-a)+c+(b-a)=c(b-a+1)$ .
- (46) 52. Given  $a<b<c$ , then  $(c-1)(b-a)+(a-1)(c-b)+c+(b-a)=b(c-a+1)$ .
- (47-8) 53. Find  $x,y,z$  such that  $x+(y+z)/a=y+(x+z)/b=z+(x+y)/c$ , where  $a<b<c\dots$
- (49) 54. Find  $x$ , such that fractions of  $x$  minus smaller fractions of  $x$ , equals  $a$ .

- (50) 55. Given that  $x + \text{big fractions of } x$  is smaller than  $y + \text{small fractions of } y$ , find  $z$ , such that  $x + \text{big fractions of } z+y$ , equals  $y + \text{small fractions of } z+x$ .
- (51) 56. Find  $x$ , such that given fractions of  $x$ , equal other given fractions of  $a$ .
- (52) 57. Find  $x,y$ , such that  $x + \text{fractions of } y$ , equals  $y + \text{other fractions of } x$ .
- (53) 58. Find  $x,y,z$ , such that  $x+z = ay$  and  $y+z = bx$ .
- (54) 59.  $(a_1)^2(a_2)^2(a_3)^2 \dots (a_n)^2 = ((a_1)(a_2)(a_3) \dots (a_n))^2$
- (55)  $(ab^2)(ac^2) = (abc)^2$
- (56a) 60.  $(a_1)^3(a_2)^3(a_3)^3 \dots (a_n)^3 = ((a_1)(a_2)(a_3) \dots (a_n))^3$
- (56b)  $(ab^3)(a^2c^3) = (abc)^3$
- (57) 61.  $ab^2+ba^2 = ab(a+b)$
- (58) 62.  $(a+b)^3 = a^3+3ab(a+b)+b^3$
- (59) 63.  $P_{n+1} = (n+1)P_n$ , where  $P_n$  is the number of different ways to order  $n$  elements.  
(Corollary:  $P_n = n!$ ).
- (60) 64.  $P_{n,2} = n(n-1)$ , where  $P_{n,m}$  is the number of ways to order  $m$  elements out of  $n$ .
- (61) 65.  $P_{n,m+1} = P_{n,m} (n-m)$ .  
(Corollary:  $P_{n,m} = n!/(n-m)! = n(n-1)(n-2) \dots (n-(m+1))$ ).
- (62) 66.  $P_{n,m} = C_{n,m}P_m$ , where  $C_{n,m}$  is the number of ways to choose  $m$  elements out of  $n$  without regard to order.
- (63) 67.  $C_{n,m} = P_{n,m}/P_m$
- (64) 68.  $C_{n,n-m} = C_{n,m}$ .

## References

1. Berggren, J.L. Episodes in the Mathematics of Medieval Islam. New York: Springer-Verlag, 1986.
2. Berggren, J.L. Mathematics and Her Sisters in Medieval Islam: A Selective Review of Work Done from 1985 to 1995. *Historia Mathematica*, **24**, (1997), pp. 407-440.
3. Bleich, J. David, *Providence in the Philosophy of Gersonides*, Studies in Judaica, New York: Yeshiva University Press, 1973.
4. Busard, H. L. L., *Jordanus de Nemore: De Elementis Arithmetice Artis, A Medieval Treatise on Number Theory, Part I: Text and Paraphrase*, Stuttgart: Franz Steiner Verlag, 1991.
5. Carlebach, Joseph, *Levi ben Gerson als Mathematiker*, Berlin: J. Lamm, 1910.
6. Curtze, Maximilian, Die Abhandlung des Levi ben Gerson ueber Trigonometrie und den Jacobstab, *Bibliotheca Mathematica* **12**, (1898), 97-112.
7. Chemla, Karine and Pahaut, Serge, Remarques sur les Ouvrages Mathematiques de Gersonide, in *Studies on Gersonides-A 14th Century Jewish Philosopher-Scientist*, ed. Gad Freudenthal, Leiden: Brill, (1992), pp. 149-191.
8. Davidson, Israel, Levi ben Abraham ben Hayim, A Mathematician of the XIIIth century, *Scripta Mathematica* **6**, (1936), 57-65.
9. Encyclopedia Judaica, Volume 11, Macmillan and Company, 1971, pp. 91-94.
10. Espenshade, P. H., A text on trigonometry by Levi ben Gerson (1288-1344), *Mathematics Teacher* **60**, (1967), 628-637.
11. Goldstein, B. R., *The Astronomy of Levi ben Gerson,(1288-1344): A Critical Edition of Chapters 1-20 with Translation and Commentary*, New York: Springer-Verlag, 1985.
12. Goldstein, B. R., Levi ben Gerson's Contributions to Astronomy, in *Studies on Gersonides-A 14th Century Jewish Philosopher-Scientist*, ed. Gad Freudenthal, Leiden: Brill, (1992), pp. 3-19.
13. Glasner, Ruth, Levi ben Gershom and the Study of Ibn Rushd in the Fourteenth Century, *The Jewish Quarterly Review*, **LXXXVI**, Nos. 1-2, (1995), pp. 51-90.
14. Hughes, Barnabas B., *Jordanus de Nemore: De numeris datis*, University of California Press, Berkeley, 1981.
15. Kasir, D., *The Algebra of Omar Khayyam*, New York: Columbia University, 1931.
16. Katz, Victor J., *A History of Mathematics: An Introduction*, New York: Harper Collins, 1993.
17. Katz, Victor J., Combinatorics and Induction in Medieval Hebrew and Islamic Mathematics, in *Vita Mathematica: Historical Research and Integration with Teaching*, ed. Ron Calinger, Washington DC: Mathematical Association of America, (1996), pp. 99-107.

18. Kellner, Menachem, An Annotated List of Writings by and about R. Levi ben Gershom, in *Studies on Gersonides-A 14th Century Jewish Philosopher-Scientist*, ed. Gad Freudenthal, Leiden: Brill, (1992), pp. 367-414.
19. Kool, Robert, Coin Department, Israel Antiquities Authority, personal communication.
20. Lange, Gerson, *Sefer Maassei Chosheb-Die Praxis des Rechners, Ein hebraeisch arithmetisches Werk des Levi ben Gerschom aus dem Jahre 1321*, Frankfurt am Main: Louis Golde, 1909.
21. Langermann, Y. Tzvi and Simonson, Shai, The Hebrew Mathematical Tradition, for *Mathematics Across Cultures: A History of Western Mathematics*, editor Helaine Selin, Kluwer Publishing, to appear in 2000.
22. Levey, M., *The Algebra of Abu Kamil, in a Commentary by Mordechai Finzi. Hebrew Text, Translation and Commentary with Special Reference to the Arabic Text*, The University of Wisconsin Press, Madison-Milwaukee, and London, 1966.
23. Levy, Tony, Gersonide, le Pseudo-Tusi et le Postulat des Paralleles, *Arabic Science and Philosophy* **2**, (1992), 39-82.
24. Manekin, C., *The Logic of Gersonides*, Kluwer, (1992).
25. Phillips, Marcus, The Gros Tournois in the Mediterranean, in *Proc. of 14<sup>th</sup> Oxford Symp. on Coinage and Monetary History*, ed. N. J. Mayhew, (1997), pp. 279-301.
26. Rabinovitch, Nahum L., Rabbi Levi ben Gershon and the Origins of Mathematical Induction, *Archive for History of Exact Sciences* **6**, (1970), 237-248.
27. Rashed, R., *The development of Arabic mathematics: between arithmetic and algebra*. Dordrecht: Kluwer. 1994.
28. Rebstock, U., Der Mu`amalat-Traktat des Ibn al-Haytham, *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften* **10** (1995-6), pp. 61-121.
29. Sarton, George, *Introduction to the History of Science*, Vol. III, pt.1, Washington: Carnegie Institute, (1947), pp. 594-607.
30. Silverberg, Moshe, *Sefer Hamispar of Rabbi Abraham ibn Ezra*, Frankfurt am Main: Y. Kaufmann, 1895.
31. Simonson, Shai, Mathematical Gems of Levi ben Gershon, *Mathematics Teacher*, to appear in 2000.
32. Simonson, Shai, The Mathematics of Levi ben Gershon, *Bekhol Derakhekha Daehu* **10**, Bar-Ilan University Press, 1999.
33. Singmaster, David, Puzzles from the Greek Anthology, *Math. Spectrum* **17:1**, (1995), pp. 11-15.
34. Sperber, D., Islamic Metrology from Jewish Sources, *Numismatic Chronicle* **7<sup>th</sup> Series**, (1972), pp. 275-282.
35. Spufford, Peter, *Money and Its Use in Medieval Europe*, Cambridge, 1988.
36. Woepcke, F., *Études sur des mathématiques arabo-islamiques. Nachdruck von Schriften aus den Jahren 1842-1874. 2 vols.*, Frankfurt: Institut für Geschichte der arabisch-islamischen Wissenschaften, 1986.

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