A Rabbi, Three Sums, and Three Problems

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1   Introduction

We present a slice of discrete math history and connect it to three neat problems and their solutions. The solutions emphasize the value of exploring and tabulating data in order to help discover theorems. Often, the exploration not only helps discover theorems, but suggests a direction and idea for a proof.

The methodology is a model of how to use mathematical history and data patterns in teaching discrete mathematics. Proofs in discrete math tend to be constructive. Discovering patterns can suggest an algorithm, that in turn can suggest a proof. Furthermore, many theorems in discrete math make use of basic sums and combinatorial identities that have been known for hundreds of years. Familiarity with the history of an identity can also suggest relevant proof methods.

Underlying the content in this article is the implicit theme of how mathematics and computers support each other. Computers can be used to generate data for pattern searching, or to help with calculations that are otherwise too cumbersome. Symbiotically, these same theorems, whose statements and proofs are motivated by the data gathered with the use of a computer, are themselves applicable to computer science in the analysis of algorithms as well as other areas.

2   A Rabbi's Identities

Rabbi Levi ben Gershon (1288–1344) lived in southern France and was a well-known scientist, mathematician, and philosopher among both Jewish and Christian scholars of his day [Si1], [Si2]. Levi discovered formulas for certain kinds of sums. These include the well-known formulas for sums of consecutive integers, sums of consecutive squares, and sums of consecutive cubes.

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{A}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{B}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{C}$$

Levi was certainly not the first to discover these formulas, but he was the first to use mathematical induction style arguments in his proofs [Ka]. Levi used induction to prove (C), but different methods for (A) and (B). His methods serve as a good review of techniques for generating and proving these closed formulas, as well as a source of clever exercises for students.

Levi's work also highlights the pros and cons of symbolic algebra versus the Euclid-style "prose"-algebra of the middle ages. In symbolic algebra we write $x^2 + y^2$, but for Euclid, Levi, and all mathematicians before the 16th century, this formula would have been written out as "the sum of the square of x with the square of y." Did the limitations of "prose"-algebra prevent Levi from extending his induction proofs to other sums? Perhaps Levi simply

modeled each proof on whatever constructive ideas led him to the discovery of the sum. No one knows the answers to these questions, but they make for good debate in the classroom. Levi's work is not widely known, and we hope the reader enjoys the details of this discussion.

Each of Levi's three formulas (A), (B), and (C) plays a role in the solution of three different problems discussed in Section 3.

2.1   Triangle Numbers

The numbers generated by formula (A) are called *triangle numbers*, because they look like triangles if you draw them with dots. Triangle numbers have a natural combinatorial interpretation. The nth triangle number is equal to the number of ways to choose two items from $n + 1$. This is usually written as $\binom{n+1}{2}$. If you try to count these pairs and number the items 1 through $n + 1$, then the first item can be paired with n others, the second item with $n - 1$ others, and so on, until the $(n - 1)$st item can be paired with only the nth item. This sum is $1 + 2 + \cdots + n$. The first five triangle numbers are 1, 3, 6, 10, and 15.

Although Levi understood the distributive property, and he could essentially perform algebraic manipulations like "FOIL"-ing, he had no symbolic algebra as we know it today. All his work was written out in Euclid-style prose just like the work of his Jewish, Christian, and Islamic contemporaries. Without symbolic algebra, we lose the compact form of formula (A) but we gain three new ways to read that formula. It is a lesson worth remembering.

Indeed Levi has three versions of formula (A) each with its own proof. These are theorems (26), (27), and (28) in his book *Maaseh Hoshev*, literally "the way of calculation." The title of his book is also a pun on a biblical phrase that describes labor on the tabernacle needing intensive thought, analysis, and calculation.

> (26) The sum of consecutive numbers from one up to an even number is equal to half the number of terms times the number of terms plus one.

Levi proves this identity by adding the numbers up in pairs, each successive pair coming from the front and back of the series. He notes that each pair sums to $n + 1$ and there are $n/2$ such pairs. He had two other proofs to take care of the case when n is odd.

> (27) The sum of consecutive terms from one up to an odd number is equal to the middle term times the number of terms.

The proof pairs up numbers from the inside working outward, showing that each pair sums to twice the middle term.

> (28) The sum of consecutive numbers from one up to an odd number is equal to half the last term times the number following the last term.

The proof uses (27) with some additional algebra arguments.

2.2   Square Pyramid Numbers

A number generated by formula (B) is called a *square pyramid number*, because it can be constructed by stacking squares of diminishing size on top of one another, forming a square-base pyramid. The first five square pyramid numbers are 1, 5, 14, 30, and 55. Levi's proof of this formula is longwinded and complex, building the formula logically from many simpler algebraic identities. A sketch of his proof with many details omitted is shown below. Filling in the missing details for each step is a good source of exercises for students.

The numbering of the lemmas is faithful to his book *Maaseh Hoshev*. Going forward, all the lemmas are translated from Levi's prose into modern symbolic notation.

First he proves the two identities, (29) and (30), by "pairing" tricks. In the first identity (29), he pairs numbers from back and front as he did for triangle numbers. In the second identity (30), he pairs 1 from the left group with n from the right, 2 from the left group with $n - 1$ from the right, etc.

$$(1 + 3 + \cdots + (2n - 1)) = n^2, \tag{29}$$

$$(1 + 2 + \cdots + n) + (1 + 2 + \cdots + n + (n + 1)) = (n + 1)^2. \tag{30}$$

Using (30) Levi proves

$$1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + \cdots + n) = \begin{cases} 2^2 + 4^2 + 6^2 + \cdots + n^2 & \text{if } n \text{ is even,} \\ 1^2 + 3^2 + 5^2 + \cdots + n^2 & \text{if } n \text{ is odd.} \end{cases} \quad (32)$$

He puts lemma (32) aside for later. Then he proves three more simple lemmas.
He proves

$$(1 + 2 + 3 + \cdots + n) + (2 + 3 + 4 + \cdots + n) + \cdots + n = 1^2 + 2^2 + 3^2 + \cdots + n^2 \quad (33)$$

by rearranging the numbers.
He proves

$$\begin{aligned} (1 + 2 + 3 + \cdots + n) &+ (2 + 3 + 4 + \cdots + n) + \cdots + n \\ &+ 1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + \cdots + (n - 1)) \\ &= n(1 + 2 + 3 + \cdots + n) \end{aligned} \quad (34)$$

not by using (33) but by another simple rearranging argument.
And, he proves

$$(n + 1)^2 + n^2 - (n + 1 + n) = 2n^2 \quad (35)$$

by essentially multiplying it out.
Levi then uses (33) and (35) to prove

$$\begin{aligned} (1 + 2 + 3 + \cdots + n) &+ (2 + 3 + 4 + \cdots + n) + \cdots + n - (1 + 2 + 3 + \cdots + n) \quad (36) \\ &= \begin{cases} 2(2^2 + 4^2 + 6^2 + \ldots + (n - 1)^2) & \text{when } n - 1 \text{ is even,} \\ 2(1^2 + 3^2 + 5^2 + \cdots + (n - 1)^2) & \text{when } n - 1 \text{ is odd.} \end{cases} \end{aligned}$$

And, he combines (32), (34), and (36) to prove

$$n(1 + 2 + 3 + \cdots + (n + 1)) = \begin{cases} 3(1^2 + 3^2 + 5^2 + \cdots + n^2) & \text{when } n \text{ is odd,} \\ 3(2^2 + 4^2 + 6^2 + \cdots + n^2) & \text{when } n \text{ is even.} \end{cases} \quad (37)$$

Finally, Levi uses (32), (33), (34), and (37) to prove

$$\left(n - \frac{n - 1}{3}\right)(1 + 2 + 3 + \cdots + n) = 1^2 + 2^2 + 3^2 + \cdots + n^2. \quad (38)$$

This last identity is formula (B) in disguise. Notice that formula (A) can be used to replace $(1 + 2 + 3 + \cdots + n)$ with $\frac{n(n+1)}{2}$.

2.3   Sums of Cubes

Levi proves formula (C) by mathematical induction. The proof by induction for formula (C) comes directly from the inductive lemma

$$(1 + 2 + 3 + \cdots + n)^2 = n^3 + (1 + 2 + 3 + \cdots + (n - 1))^2 \quad (41)$$

which he proves essentially by "FOIL"-ing out the left side. The proof of (C) is remarkably short and clear in comparison to the proof of (B).

Levi was one of the first to use the idea of mathematical induction, 300 years before Pascal [Ka]. It is interesting that he chose induction to prove formula (C), but not for formulas (A) and (B). Why do you think this was? His proof for (A) is fairly simple, but considering the complexity of his proof for (B), one naturally wonders why he did not use induction there as well. These are the kinds of questions that interest historians of mathematics.

3   Three Problems

Building on Levi's work, we tackle three counting problems. The first two are fairly well-known and have elegant closed-form solutions. The third problem is not well-known and the solution requires extra effort. Before we start on the three problems it is helpful to solve a little warm-up problem, and add one more formula, (D), to Levi's (A), (B), and (C).

3.1   Triangular Pyramid Numbers — A Warm-Up Problem

The sum of the first n triangle numbers is called a *triangular pyramid number* because it is formed by stacking triangles of diminishing size on top of each other into a triangular pyramid. Let's warm up by deriving a closed form for the nth triangular pyramid number. The easy way to do this is to use Levi's formulas (A) and (B) to help:

$$\sum_{i=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2}\left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right) = \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right).$$

Simplifying the right side proves that the nth triangular pyramid number is equal to

$$1 + 3 + 6 + \cdots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}. \tag{D}$$

With formulas (A), (B), (C), and (D), we are armed to attack our three counting problems. However, before we move ahead to the three problems promised, there is a short interesting tangent showing a connection between the relatively simple discrete math formulas (B) and (D), and state of the art number theory.

Is there a number (besides 1) that is both a triangular pyramid number and a square pyramid number? That is, do formulas (B) and (D) generate any common integers besides 1? It would make a good children's Sherlock Holmes mystery, where somebody knocks down a pyramid of oranges in the store, and when the oranges are rebuilt into a pyramid, it looks like none are missing, but the grocer insists that some oranges are missing. A young Holmes interviews the grocer and confirms that the original pyramid was square while the rebuilt pyramid is triangular. Holmes declares, "The grocer is telling the truth; there are oranges missing. Because as everyone knows, there is no number besides 1 that is both a triangular pyramid number and a square pyramid number."

Well, in fact, nobody knew this for sure until 1988 when Beukers and Top proved it using some high powered methods [BT], [HL]. Their work is a special case of finding rational points on elliptic curves.

Now let's begin on the three problems.

3.2   Problem 1 — Counting Squares in a Grid

How many squares are there in an n by n grid? See Figure 1.
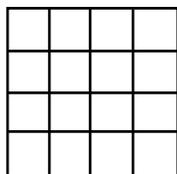
Figure 1. A 4 by 4 grid.

Before one tackles any counting problem, it never hurts to get some data. Let's make a chart for the first four values of n, and count the squares explicitly. You can generate the first few cases by hand pretty quickly, but if you are a clever programmer, you might enjoy writing a program to count the squares. The advantage of a program is that once it's finished, it generates lots more data very quickly. Moreover, sometimes the process itself of writing a program reveals an organized way to think about the counting. We won't need a program in these examples, but the use of programs in similar problems can be exciting and helpful for students and researchers alike.

n	Number of Squares in an n by n Grid
1	1
2	5
3	14
4	30

Compare the numbers in the table above with those generated from identity (B), and observe that they are identical. Indeed, it looks as though the number of squares in an n by n grid is the same as the sum of the first n squares. And, we can prove it. There are n^2 squares of size 1, and $(n-1)^2$ squares of size 2, $\ldots$, and one square of size n.

3.3   Problem 2 — Counting Rectangles in a Grid

Counting the number of rectangles explicitly is a lot more work than counting squares. Doing it for grids of size 1 by 1, 2 by 2, 3 by 3, and 4 by 4, we get the chart below:

n	Number of Rectangles in an n by n Grid
1	1
2	9
3	36
4	100

The pattern is interesting. The numbers are squares, and moreover they are squares of triangle numbers. Levi's identity (C) says that the square of the nth triangle number is the sum of the first n cubes. It seems, therefore, that the number of rectangles in an n by n grid is the sum of the first n cubes.

We can construct a combinatorial proof of this discovery. Indeed, attempting to write a program to count the rectangles might lead to this proof. Each rectangle in an n by n grid is uniquely defined by selecting two of the $n+1$ vertical lines and two of the $n+1$ horizontal lines. The area outlined by these four lines is the rectangle that they define. Hence the number of rectangles is

$$\binom{n+1}{2}^2 = (1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3.$$

3.4   Problem 3 — Counting Triangles

How many triangles in the triangular "grid" with four edges per side? See Figure 2.
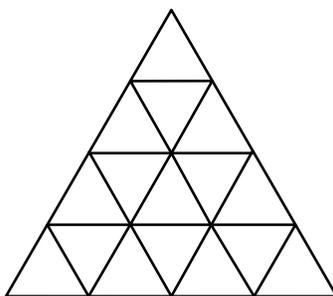
Figure 2. A triangular grid.

This last puzzle does not yield its secret so easily. Our exploration builds conjectures, looks for patterns, tests the conjectures and revises them.

First we gather some data by counting the number of triangles explicitly for different size triangular figures with the number of edges on a side varying from 1 to 5. To make sure we don't miss any triangles, it is a good idea to organize

the counting by the size of the triangles, or by whether or not the triangles point up or down. The total results are summarized below:

n	Number of Triangles in a Triangular Figure with Sides of Length n
1	1
2	5
3	13
4	27
5	48

These numbers are not so familiar. We aren't as lucky as we were last time, but if we look harder we may still see a pattern that may guide to us a closed form and a proof. Let's count the triangles by size. That idea worked for squares and rectangles.

	Number of Triangles in a Triangular Figure with Sides of Length n				
n	Size 1	Size 2	Size 3	Size 4	Size 5
1	1	0	0	0	0
2	4	1	0	0	0
3	9	3	1	0	0
4	16	7	3	1	0
5	25	13	6	3	1

This table has one nice column, the square numbers of column 1, but column 2 is not familiar, and the other columns are too short to reveal any patterns. Let's explore further. You may have noticed that some of the triangles point upward and some point downward. Let's separately count triangles that point up, and triangles that point down.

	Number of Triangles in a Triangular Figure with Sides of Length n		
n	Triangles that Point Up	Triangles that Point Down	Total
1	1	0	1
2	4	1	5
3	10	3	13
4	20	7	27
5	35	13	48

The first column should look familiar. The numbers in it are the triangular pyramid numbers. However, the second column is still a mystery. Let's break it down a little more, this time by type of triangle (up or down) and size.

	Number of Triangles in a Triangular Figure with Sides of Length n										
	Triangles that Point Up					Triangles that Point Down					
n	Size 1	Size 2	Size 3	Size 4	Size 5	Size 1	Size 2	Size 3	Size 4	Size 5	Total
1	1	0	0	0	0	0	0	0	0	0	1
2	3	1	0	0	0	1	0	0	0	0	5
3	6	3	1	0	0	3	0	0	0	0	13
4	10	6	3	1	0	6	1	0	0	0	27
5	15	10	6	3	1	10	3	0	0	0	48

Now we are getting somewhere. It's a table of triangle numbers! Let T_n be the nth triangle number. The sum of the nth row of the triangles that point up is T_n larger than the sum of the $(n-1)$st row of triangles that point up. Less obvious is that the sum of the nth row of the triangles that point down is T_{n-1} larger than the sum of the $(n-2)$nd row of triangles that point down. This means we would expect the next two rows in the table to be

| 6 | 21 | 15 | 10 | 6 | 3 | 1 | 0 | 15 | 6 | 1 | 0 | 0 | 0 | 0 |
| 7 | 28 | 21 | 15 | 10 | 6 | 3 | 1 | 21 | 10 | 3 | 0 | 0 | 0 | 0 |

Once we discover patterns like these, we are directed by our own exploration and serendipitous discovery to try to explain why they might be so. Let U_n and D_n, respectively, be the number of up-pointing and down-pointing triangles in a triangular "grid" with n edges per side. Figure 2 shows the grid when $n = 4$.

Lemma 1. $U_n = U_{n-1} + T_n$.

Proof. A triangle figure with n edges per side contains all the up-pointing triangles of a similar figure with $n-1$ edges per side, plus whatever extra up-pointing triangles we get by including one or more of the n edges from the bottom of the triangle. There is one up-pointing triangle that uses all these edges (the whole figure), two that use $n-1$ of these edges, three that use $n-2$ of these edges, ..., and n triangles that use exactly one of these edges. This sum is T_n. See Figure 3 for an example when $n = 5$. □

The next proof is similar but slightly more difficult than the previous one. The difficulty is surmountable because our patterns tell us exactly what we must look for in the proof. And, when you know what to look for, then a proof is easier to find.
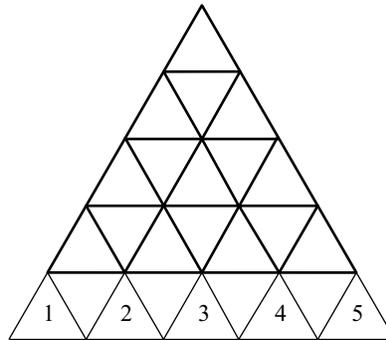
Figure 3. A triangular grid with a new row.

Lemma 2. $D_n = D_{n-2} + T_{n-1}$.

Proof. A triangle figure with n edges per side contains all the down-pointing triangles of a similar figure with $n - 2$ edges per side, plus whatever extra down-pointing triangles we get by including a point on any one of the two new rows. The number of down-pointing triangles with i edges per side that include a point on the bottom (nth) row equals $n - (2i - 1)$. For example, there are $n - 1$ down-pointing triangles with one edge per side that use points on the bottom row. This is because every time you add an edge to the side of the triangle, you lose the triangles that used the right and leftmost points on the nth row. See Figure 4 for an example when $n = 6$. For a similar reason, the number of
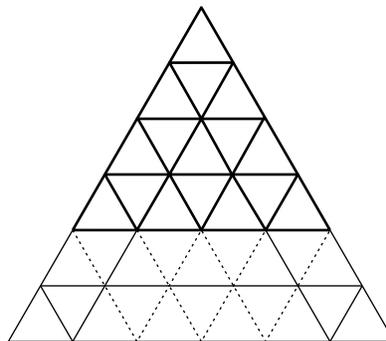
Figure 4. A triangular grid with two new rows.

down-pointing triangles with i edges per side that include a point on the next to bottom $((n-1)\text{st})$ row equals $n-2i$. The sum of these two as i ranges from 1 to $\frac{n}{2}$ is T_{n-1} (when i is greater than $\frac{n}{2}$, there are no triangles). $\qquad\square$

Let $M_n = U_n + D_n$, i.e., the total number of triangles in a triangular grid with n edges per side, (M is the transliterated first letter of the Hebrew word for triangle.) Lemmas 1 and 2 imply that $M_n = M_{n-2} + T_n + 2T_{n-1}$. Also, $M_0 = 0$ and $M_1 = 1$. Recall that $T_n = \frac{n(n+1)}{2}$. Then we have

Theorem 3.

$$M_n = M_{n-2} + \frac{n(n+1)}{2} + 2\frac{(n-1)n}{2} = M_{n-2} + \frac{3n^2}{2} - \frac{n}{2}.$$

This seems to solve our problem but it is not as satisfying as the solutions for squares and rectangles. It would be nice to get a *closed* formula for the number of triangles; that is, a formula without any recurrence or loops, like we had for squares and rectangles.

Recurrence equations are easier to discover than closed formulas, but closed formulas are often easier to use for calculations. Fortunately, many recurrence equations can be solved and turned into closed formulas by simply "unraveling" the formulas. This means repeatedly applying the recurrence over and over again, starting with the base case, constructing a summation that we then turn into a closed form. In this case, since the recurrence equation skips two indices with each iteration, we get two different summations, one for the even indices and one for the odd.

Starting with even indices, we have

$$M_0 = 0,$$
$$M_2 = M_0 + \frac{3}{2}\left(2^2\right) - \frac{1}{2}\left(2\right),$$
$$M_4 = M_2 + \frac{3}{2}\left(4^2\right) - \frac{1}{2}\left(4\right).$$

This generalizes to

$$
\begin{aligned}
M_{2n} &= \frac{3}{2}\left(2^2 + 4^2 + 6^2 + \cdots + (2n)^2\right) - \frac{1}{2}\left(2 + 4 + 6 + \cdots + 2n\right) \\
&= \frac{3}{2} \times 4\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) - \left(1 + 2 + 3 + \cdots + n\right) \\
&= 6\frac{(n(n+1)(2n+1))}{6} - \frac{n(n+1)}{2} \\
&= n(n+1)(2n+1) - \frac{n(n+1)}{2} \\
&= \frac{(4n^3 + 5n^2 + n)}{2}.
\end{aligned}
$$

The first line above comes from unraveling the recurrence of Theorem 3.3; the third line is derived from the second line using Levi's identities (A) and (B); the fourth line is due to identity (A); and the rest is straightforward algebra.

As we did for the even indices, let's find the pattern and closed form for the odd values, M_{2n+1}.

$$
\begin{aligned}
M_{2n+1} &= 1 + \frac{3}{2}\left(3^2 + 5^2 + \cdots + (2n+1)^2\right) - \frac{1}{2}\left(3 + 5 + \cdots + (2n+1)\right) \\
&= \frac{3}{2}\left(1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2\right) - \frac{1}{2}\left(1 + 3 + 5 + \cdots + (2n+1)\right) \\
&= \frac{(2n+1)\left(1 + 2 + 3 + \cdots + (2n+2)\right)}{2} - \frac{(n+1)^2}{2} \\
&= \frac{(2n+1)(n+1)(2n+3)}{2} - \frac{(n+1)^2}{2} \\
&= \frac{4n^3 + 11n^2 + 9n + 2}{2}.
\end{aligned}
$$

As before, the first line above comes from unraveling Theorem 3.3. The third line is derived from the second line using Levi's identities (29) and (37). And, as before, the fourth line is due to identity (A), and the rest is straightforward algebra.

The two formulas for the number of triangles in a grid are not as pretty as the closed forms for the number of squares and rectangles. But they are the truth, and you can't choose the truth! An equivalent but much prettier way to write these formulas is

$$M_n = \begin{cases} \frac{n(n+2)(2n+1)}{8} & \text{when } n \text{ is even,} \\\\ \frac{n(n+2)(2n+1)-1}{8} & \text{when } n \text{ is odd.} \end{cases}$$

or simply

$$M_n = \frac{n(n+2)(2n+1)}{8} + \frac{1}{16}\left((-1)^n - 1\right).$$

The stuff at the end of the formula just takes care of the minor difference between the odd and even cases. Notice the similarity between the main part of this formula $\frac{n(n+2)(2n+1)}{8}$ and the closed form for the number of squares in a grid, $\frac{n(n+1)(2n+1)}{6}$.

4   Conclusion

We presented a trio of problems that focus on the benefits of discovery and experiment in discrete mathematics. The solutions to the problems use recurrence equations, summations, and counting concepts, and involve some famous identities.

There are many ways this material can be used in the classroom. Several identities of Levi ben Gershon (1288–1344) can be used with discussion centered on contrasting different style proofs and analyzing why someone might use one technique over another. The three problems (squares, rectangles and triangles in a grid) can be used in group exploration and discovery, with the instructor as the guide in a Polya-style lecture.

5   Acknowledgements

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