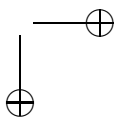
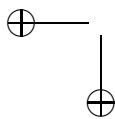
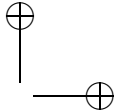
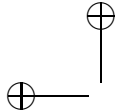


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# A Guide for the Reader

What does a mathematician do? Most people know that scientists design experiments to verify theories of how the world works; writers write to communicate, educate, or evoke emotions and thoughts; politicians influence policy to hopefully make our lives better; and musicians perform or compose to bring joy and meaning to their lives and to others. But mathematicians? Few people have any notion of what a mathematician does day to day. Many people associate mathematics with arithmetic, terminology, memorization, drills and methods, but there is much more to mathematics.

If you asked a mathematician, you might be surprised to hear her say that what she does for a living feels closer to writing poetry than doing rote calculations. Indeed, mathematics is more about planning, exploring, creating, and experimenting than it is about memorizing and labeling. It is elegant, logical, beautiful and inspiring. Mathematicians solve problems; they search for structure and truth; they seek to understand why shapes, ideas, numbers and patterns interact and behave the way they do.

## What is this Book About?

The underlying theme of this book is that mathematics education should focus less on rote memorization of terminology and algorithms, and more on understanding and deep comprehension that comes through investigation, experiment, and personal exploration. This book is about rediscovering mathematics. Most of you have already discovered math—the formulas and terminology learned in school—but few have had the joy of the wider world of mathematics, filled with creativity, exploration, experiment, and variety. Indeed, I hope this book allows you to rediscover mathematics with the maturity, perspective and patience that come with distancing yourself from your past experiences.

The book does not survey all of mathematics, nor does it concentrate on just one special area. It is a flight over the landscape of mathematics with occasional stops at places that look interesting. Topics include well-known puzzles, obscure ones, and a few published here for the first time. Each puzzle is focused on a particular theme, intended to challenge the reader to discover its secrets. *Challenges* are presented as the secrets are slowly discovered and the puzzles are unraveled. You are given the opportunity to try to figure out the next part of a puzzle yourself, or just read the solution and continue. The challenges are

meant to entertain, educate, and motivate. Along the way, there is advice on how to study, read and teach math. Both specific mathematical topics and problem-solving techniques are presented. With every challenge comes a blueprint for how an experienced mathematician *might* have solved it. For further practice and exploration, there are additional challenges at the end of each chapter, without the solutions provided.

## Is this Book for You?

Are you a teacher? This book will help you make mathematics come alive for your students. Your students will understand more, memorize less, and enjoy the *process* of discovering mathematics. Depending on the level of your students, some lesson plans can come directly from your reading, but even when the specific mathematics is too advanced for your students, you can design your own dynamic lessons using the examples in the book as models. The effort you expend in reading this book and rediscovering mathematics will empower you to pass on your knowledge and reshape the popular perception of mathematics—one child at a time.

The book is not just for teachers or prospective teachers. It is for anyone who is intellectually curious and looking for a guide to help revisit and reconsider mathematics. It is for the math whiz looking for interesting topics outside the standard fare, as well as for the math phobic willing to give mathematics another chance.

## Preliminaries

Rediscovering mathematics is not a spectator sport; you must get involved; you must be an active participant. And like any sport, in order to participate, you must first get in shape. Mathematical fundamentals need to be mastered before any discoveries can be accomplished. Getting in shape mathematically may be hard work, but the reward is personal mathematical discovery that brings joy and builds confidence.

When you first take golf lessons, you spend a lot of time training your body to move in the right way. Impatience causes some people to just go out and hit the ball, but they develop bad habits that are hard to unlearn. Golf is a complex sport with technique and strategy, and the way to best appreciate it is to study the fundamentals, and then play, and play some more. Mathematics is the same as golf: master the fundamentals and then practice their application on various challenges.

For this book, the fundamentals you are expected to know include: arithmetic of whole numbers and fractions, a little terminology, basic algebra, and elementary geometry—the same set of topics expected of you for the SAT. The challenges that appear throughout the book are your opportunity to practice. Try every challenge, but be patient. Not every challenge will yield its secret easily. The challenges vary in difficulty from the routine to the subtle. Even if you solve just a few of the challenges, your failed attempts will not be in vain.

It took Andrew Wiles eight years to complete his proof of Fermat’s Last Theorem, a problem that had baffled mathematicians for centuries. Indeed, after seven years, Wiles presented a “proof” at an international conference only to later find a flaw. Wiles, along with Richard Taylor, took another year to untangle the error. You won’t need eight years to

solve any challenge presented here, but there will be times when you get stuck. Often, you may not be able to discern how a particular technique or skill helps you solve a problem. Have faith in yourself and in the subject. “Wax on, wax off,” Mr. Miagi tells his impatient young student in the movie *The Karate Kid*. The boy complains that he wants to start learning karate and not just wax cars. Later he learns that the motion he has used over and over to wax the cars has taught him the instinctive motion to block an attack. For Andrew Wiles, it was precisely an earlier failed effort that showed him the way to patch up the flaw in his proof of Fermat’s Last Theorem.<sup>1</sup> He summarized his long battle with Fermat’s Last Theorem in this way: “However impenetrable it seems, if you don’t try it, then you can never do it.” Patience.

## Why Bother?

Math can be hard, the fundamentals challenging, and other pursuits are easier and more fun. So why bother? The answer is because mathematics is an intrinsically beautiful subject. The study and practice of mathematics can raise your spirits, gladden your heart, and put a smile on your face. However, even if your appreciation of the subject never rises to a level of passion, be assured that studying mathematics makes you sharper. Exploring mathematical challenges is the intellectual equivalent of physical conditioning. After working through this book, you will analyze more quickly, think more critically, and ponder things you never used to even notice. You will become someone not easily fooled.

## How to Use this Book

Try hard *not* to read the solutions to the challenges until you have explored, investigated, and tried to solve them on your own. In a classroom setting, I have students work in groups on a challenge, and follow up by discussing alternative approaches and evaluating each group’s progress and effectiveness. We review “solutions” only after each group has submitted a journal of their attempts, successes, failures, and discoveries.

There are many ways to attack each challenge in this book, and the “solutions” provided represent just one option. You may find that you have a more elegant insight to a challenge or a better solution. Trust yourself—you probably do. When I teach using this book, the class often discovers new and interesting approaches that I had not considered. Finally, solutions to challenges almost always involve writing, rather than a simple “answer” like an equation or a number. You must be convincing, logical, organized, and rigorous. Learning to write effectively and clearly is as important in mathematics as it is in every academic endeavor. See the discussion in Chapter 3 about proofs and writing mathematics.

This is the sort of book that should *not* be read in one sitting. Keep it on your night table with a pad of paper nearby, and work through it at your leisure. I recommend that you start by reading the Introduction about how to read mathematics. After that, feel free to skip around from chapter to chapter. The chapters are mostly independent, and sticking to what you like is more important than any linear overarching structure. The only exception is that Chapter 10 depends slightly on the material in Chapter 3.

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<sup>1</sup>NOVA Online: *The Proof*, WGBH, 1997.

All the chapters mix technical material with pedagogical advice, but Chapters 1, 5, 9, 12, and 13 have a more pedagogical focus than the other chapters. Chapters 1 and 13 respectively introduce and revisit the main themes of the book. Chapters 4 and 7 are the most mathematically difficult chapters, and are appropriate for more ambitious readers. If you are looking for particular mathematical topics, the following list will help you choose a chapter.

Probability	Introduction and Chapter 10
Algebra	Chapters 2, 3, 4, 5, 6, 7, 9, 10, and 11
Geometry	Chapters 5, 6, 8, 9, and 12
Number Theory	Chapters 1, 2, 3, 4, and 7
Algorithms	Chapters 1, 2, 3, and 7
Combinatorics	Chapters 4, 10, and 11

The book can be picked up again and again, each time providing you with a new experience. If you get frustrated or tired, put the book down and get some ice cream. Come back again when you are happy and ready for a new challenge.

— SS  
Sharon, MA